One- and Two-Photon Wave Mechanics

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OUTLINE

• States versus Modes

• Photons versus EM Fields

• Monochromatic Modes versus Temporal-Spatial (Wave-Packet) Modes
Team Maxwell-Boltzmann vs. Team Bose-Einstein

Light is made of EM waves. Modes are distinguishable. M-B statistics applies.

Light is made of corpuscles. They are indistinguishable. B-E statistics applies.

EM modes as entities. Photons as state description.

Photons as entities. Quantum field as emergent.

Planck spectrum
Paul Dirac:

“The number of bosons in any boson state is equal to the degree of excitation of the corresponding oscillator. This is a rather remarkable fact, and it forms the basis of the reconciliation of the wave and corpuscular theories of light.”

“why think about single-and double-photon wave functions?

Prior to experiments on blackbody spectrum, no one needed quantum theory of light. Then the Planck spectrum was measured. Then Planck/Bose/Einstein invented quantum theory of light.

Prior to the laser, no one needed quantum coherence theory of light. Then the laser was invented and it produced coherent states. Then Glauber invented quantum coherence theory. Because photon number was indeterminate, Glauber’s approach was based on quantum field theory (QFT). A major accomplishment was showing the connection of QFT and CL coherence theory, which are connected by models of light detection. The major connection is QF<-->CF

Prior to single and double photon sources, no one needed a theory of photon wave mechanics (PWM). Then B-B and Sipe clarified PWM. Because the photon number is small and fixed, their approach is based on photon wave functions (PWF) for photon as particles. Here we show the connection of PWM and CL coherence theory in the few-photon regime. The major connection is PWF<-->CWF
Quantum Field using Monochromatic Modes

\[
\overline{E}^{(+)}(\mathbf{r}, t) = \sum_\lambda \int d^3k \sqrt{k} \hat{a}(\mathbf{k}, \lambda) \mathbf{u}_{\mathbf{k}, \lambda}(\mathbf{r}, t)
\]

Bosonic operators:

\[
\left[ \hat{a}(\mathbf{k}, \lambda), \hat{a}^\dagger(\mathbf{k}', \alpha) \right] = \delta(\mathbf{k}, \mathbf{k}') \delta_{\lambda, \alpha}
\]

Monochromatic modes:

\[
\mathbf{u}_{\mathbf{k}, \lambda}(\mathbf{r}, t) = \mathbf{e}_\lambda \exp(ik \cdot r - i\omega t)
\]

Quantum Field using Temporal-Spatial (Wave-Packet) Modes


\[
\overline{E}^{(+)}(\mathbf{r}, t) = \sum_j \hat{b}_j \mathbf{v}_j(\mathbf{r}, t) \quad \left[ \hat{b}_j, \hat{b}_m^\dagger \right] = \delta_{j, m}
\]

Unitary transformation:

\[
\hat{b}_j = \sum_\lambda \int d^3k R_j^*(\mathbf{k}, \lambda) \hat{a}(\mathbf{k}, \lambda)
\]

Non-Monochromatic modes (wave packets):

\[
\mathbf{v}_j(\mathbf{r}, t) = \sum_\lambda \int d^3k R_j(\mathbf{k}, \lambda) \sqrt{k} \mathbf{u}_{\mathbf{k}, \lambda}(\mathbf{r}, t)
\]
Spatial Orthogonality (Dirac)

One-photon state:
\[ \hat{a}^\dagger (\vec{k}', \alpha) |\text{vac}\rangle = |1\rangle_{\vec{k}', \alpha} \]

Monochromatic modes:
\[ \vec{u}_{\vec{k}, \lambda}(r, t) = \mathcal{E}_\lambda \exp(i \vec{k} \cdot \vec{r} - i \omega t) \]

Spatial Orthogonality:
\[ \int d^3r \, \vec{u}_{\vec{k}, \lambda}(r, t)^* \vec{u}_{\vec{k}', \alpha}(r, t) = \delta(\vec{k}, \vec{k}') \delta_{\lambda, \alpha} \]

Wave-Packet Modes (Titulaer-Glauber)

Unitary transformation:
\[ \hat{a}(\vec{k}, \lambda) = \sum_j R_j(\vec{k}, \lambda) \hat{b}_j \quad , \quad \hat{b}_j = \sum_\lambda \int d^3k \, R_j^*(\vec{k}, \lambda) \hat{a}(\vec{k}, \lambda) \]

Non-Monochromatic modes:
\[ \vec{v}_j(r, t) = \sum_\lambda \int d^3k \, R_j(\vec{k}, \lambda) \sqrt{k} \, \vec{u}_{\vec{k}, \lambda}(r, t) \]

One-photon state:
\[ \hat{b}_j ^\dagger |\text{vac}\rangle = |1\rangle_j = \sum_\lambda \int d^3k \, R_j(\vec{k}, \lambda) |1\rangle_{\vec{k}', \alpha} \]

Spatially Non-Orthogonal:
\[ \int d^3r \, \vec{v}_j(r, t)^* \vec{v}_m(r, t) \neq \delta_{j, m} \]

In linear algebra we are free to invent a new scalar product.
Non-Monochromatic modes (drop the time variable):

Spatially Non-Orthogonal:

New scalar product:

It is nonlocal.

Invent a new scalar product

$$\vec{v}_j(\vec{r}) = \sum_\lambda \int d^3k \, R_j(\vec{k}, \lambda) \sqrt{k} \, \vec{e}_\lambda \exp(ik \cdot \vec{r})$$

$$\int d^3r \, \vec{v}_j(\vec{r})^* \vec{v}_m(\vec{r}) = \sum_\lambda \int d^3k \, R_j^*(\vec{k}, \lambda) \, k \, R_m(\vec{k}, \lambda) \rightarrow$$

$$\left(\vec{v}_j | \vec{v}_m\right) \equiv \sum_\lambda \int d^3k \, R_j^*(\vec{k}, \lambda) \, k' \, R_m(\vec{k}, \lambda)$$

$$= \int d^3r \int d^3r' \frac{\vec{v}_j(\vec{r})^* \vec{v}_m(\vec{r'})}{|\vec{r} - \vec{r'}|^2} = \delta_{j,m}$$

What is this Thing? It is nonlocal.

Glauber never used or proposed this. What can possibly justify its use?

We can find a comparable scalar product in the theory of the one-photon wave function.
Relativistic Quantum Mechanics

beginning from Einstein’s equation:

\[ E = \sqrt{(mc^2)^2 + (cp)^2} \]

\[ m = \text{mass}, \ p = \text{momentum} \]

- for slow particles:

\[ E \approx mc^2 + \frac{p^2}{2m} + ... \]

Einstein rest energy          Newton kinetic energy
\[ E = \sqrt{(mc^2)^2 + (cp)^2} \]

\[ \begin{align*}
\text{(Planck)} & \quad i\hbar \frac{\partial}{\partial t} \Psi = \sqrt{(mc^2)^2 + c^2(-i\hbar \vec{\nabla})^2} \Psi \\
\text{(Einstein)} & \quad m=\text{mass} \\
p=\text{momentum} \\
\text{Dirac Equation} & \quad \Psi = cm\beta \Psi - i\hbar c (\vec{\alpha} \cdot \vec{\nabla}) \Psi \\
\text{Schrödinger Equation} & \quad \Psi^{(2)} \approx -\frac{\hbar^2}{2m} \vec{\nabla}^2 \Psi^{(2)}
\end{align*} \]

Electron:
- \( m \neq 0 \)
- \( s = 1/2 \)
- \( v \sim c \)

\( v \ll c \)
What are the differences between an electron and a photon?

Electron has:
• nonzero mass
• any speed < c
• Spin = 1/2 (two possible projections along any chosen axis, +1/2, -1/2)
• -> two-component wave function $\Psi^{(2)}$
  
  obeying the Schrödinger equation:

Photon has:
• zero mass
• speed = c
• Spin = 1 (two possible projections along propagation axis, +1, -1)
• wave function obeying what equation?
\[ E = \sqrt{(mc^2)^2 + (cp)^2} \]

\[ i\hbar \frac{\partial}{\partial t} \Psi = \sqrt{(mc^2)^2 + c^2(-i\hbar \nabla)^2} \Psi \]

\[ i\hbar \frac{\partial}{\partial t} \Psi^{(2)} \equiv -\frac{\hbar^2}{2m} \nabla^2 \Psi^{(2)} \]

\[ i \frac{\partial}{\partial t} \vec{\Psi} = c \nabla \times \vec{\Psi} \]

\[ \nabla \cdot \vec{\Psi} = 0 \]

\[ \vec{\Psi} = \vec{E} + i\vec{B} \]

\[ \frac{\partial}{\partial t} \vec{B} = -c \nabla \times \vec{E} \]

\[ \frac{\partial}{\partial t} \vec{E} = c \nabla \times \vec{B} \]
**Derivation of Maxwell’s Equations**

\[ E = \sqrt{(c p)^2} \]

\[ E \psi(p, E) = c \sqrt{p \cdot p} \psi(p, E) \]

\[ \hat{O} = \sqrt{p \cdot p} = i \hat{p} \times \]

\[ \hat{O} \hat{O} \psi_T = i \hat{p} \times (i \hat{p} \times \psi_T) = (p \cdot \hat{p})\psi_T - \hat{p}(p \cdot \psi_T) = (p \cdot p)\psi_T \]

\[ E \psi_T(p, E) = c i \hat{p} \times \psi_T(p, E) \]

\[ \psi_T(r, t) = \int \int dE\ d^3p \ \delta(E - c|p|) \exp(-iEt / \hbar + i\hat{p} \cdot \hat{r} / \hbar) f(E) \tilde{\psi}_T(p, E) \]

\[ i \frac{\partial}{\partial t} \psi_T(r, t) = c \nabla \times \psi_T(r, t) \]

Momentum wave fn

\[ \tilde{\psi}(p, E) = (\tilde{\psi}_x, \tilde{\psi}_y, \tilde{\psi}_z) \]

\[ \tilde{\psi} = \tilde{\psi}_T + \tilde{\psi}_L \]

\[ \hat{p} \times \psi_L = 0, \quad \hat{p} \cdot \psi_T = 0 \]

**Photon, m=0, s=1, 3 components**

**Riemann-Silberstein vector**

\[ \frac{\partial}{\partial t} \overline{B} = -c \overline{\nabla} \times \overline{E} \]

\[ \frac{\partial}{\partial t} \overline{E} = c \overline{\nabla} \times \overline{B} \]
For a single-photon field, the complex electromagnetic field $E \pm iB$ is the quantum wave function of the photon.

$\Psi(\vec{r}) = \begin{pmatrix} E_x(\vec{r}) + \sigma iB_x(\vec{r}) \\ E_y(\vec{r}) + \sigma iB_y(\vec{r}) \\ E_z(\vec{r}) + \sigma iB_z(\vec{r}) \end{pmatrix}$

$\sigma = \pm 1$

Maxwell’s equations give the wave equation of a single photon - a quantum particle.

Maxwell, in 1862, discovered a fully relativistic, quantum mechanical theory of a single photon.
\[ \Psi = \bar{E} + i \bar{B} \]

What is the proper Scalar Product and normalization to use?

- should be bilinear
- should be Lorentz invariant


\[ (\Psi_j | \Psi_m) = \int d^3r \int d^3r' \frac{\Psi_j(r)^* \bar{\Psi}_m(r')}{|r - r'|^2} = \delta_{j,m} \]

Same as SP for wave packets

Norm: \[ (\Psi | \Psi) = \int d^3r \int d^3r' \frac{\Psi(r)^* \bar{\Psi}(r')}{|r - r'|^2} = 1 \]

No local particle density (deal with it)

The mean Energy of the photon is:

\[ (\Psi | k | \Psi) = \int d^3r \ \bar{\Psi}(r)^* \bar{\Psi}(r) = \langle k \rangle \propto \langle E \rangle \]

Is a local energy density.

Not invariant (good)

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Summary so far:

- Titulaer-Glauber wave-packet field quantization requires a strange-looking scalar product to retain orthonormality.

\[ \langle \vec{v}_j | \vec{v}_m \rangle \equiv \int d^3r \int d^3r' \frac{\overline{\vec{v}_j}(\vec{r}) \, \overline{\vec{v}_m}(\vec{r}')}{|\vec{r} - \vec{r}'|^2} = \delta_{j,m} \]

- The same scalar product arises in photon wave-function theory as a consequence of Lorentz invariance.

\[ \langle \overline{\Psi} | \overline{\Psi} \rangle = \int d^3r \int d^3r' \frac{\overline{\overline{\Psi}(\vec{r})} \, \overline{\overline{\Psi}(\vec{r}')}}{|\vec{r} - \vec{r}'|^2} = 1 \quad \overline{\Psi} = \overline{E} + i\overline{B} \]

\[ \langle \overline{\Psi} | k | \overline{\Psi} \rangle = \int d^3r \, \overline{\Psi}(\vec{r}) \, \overline{\Psi}(\vec{r}) \propto \langle E \rangle \]

\[ \overline{\Psi}(\vec{r}) \] is the probability amplitude for localizing Energy, not particle position.
Connection to one-photon detection amplitude in QFT:
(Zubairy and Scully text)

\[ A_D(\vec{r}, t) = \langle 0 | \hat{E}^{(+)}(\vec{r}, t) | 1 \text{ photon field state} \rangle \]

This obeys a 2nd-order wv eqn:

\[ \frac{\partial^2}{\partial t^2} A_D(\vec{r}, t) = c^2 \nabla^2 A_D(\vec{r}, t) \]

Photon Wave Function contains the B part too:

\[ \Psi(\vec{r}, t) = \bar{E}(\vec{r}, t) + i \bar{B}(\vec{r}, t) \]

obeys a 1st-order, vector wv eqn, which carries more information:

\[ i \frac{\partial}{\partial t} \Psi = c \nabla \times \Psi \]
The single-photon wave function is amenable to measurement:

**QUANTUM STATE TOMOGRAPHY**

A quantum state is nothing more than a collection of acquired information about a part of nature.

Parity-Inverting Sagnac Interferometer
Measuring the photon spatial Wigner function

Parity-inverting Sagnac Interferometer
Wigner Function

The average photon count rate equals:

\[ I \equiv C_1 + C_2 \cdot \int_{-\infty}^{+\infty} \psi_0(x + x') \psi_0^*(x - x') e^{-2ik_xx'} dx' \]

\[ I \equiv C_1 + C_2 \cdot W(x, k_x) \]

The Wigner Function is uniquely related to the Schrödinger Wave Function, so its measurement reveals \( \psi(x) \) not just \( |\psi(x)|^2 \)
Example: Double-Slit Interference

Far field pattern

Measure $\psi(x)$ just after slits
Measured Wigner function for ensemble of single photons passed through double slit.

\[ x = \text{position} \]
\[ p_x = \text{transverse momentum} \]

Negative regions (blue, purple) indicate nonclassical particle behavior.
Two-Photon Wave Mechanics

Two-Photon Wave Function?

\[ \tilde{\Psi} = \sum_j C_j \tilde{\psi}_j(\vec{r}_1, t?) \tilde{\phi}_j(\vec{r}_2, t?) \]
Possible Approaches: one-time or two-time wave equations

two-time wave function:

\[ \Psi(r_1,t_1; r_2, t_2) = \sum_j C_j \psi_j(r_1, t_1) \otimes \phi_j(r_2, t_2) \]

Pair of two-time Max. Eqns.

\[ i \frac{\partial}{\partial t_1} \Psi(r_1,t_1; r_2, t_2) = c \nabla_1 \times \Psi(r_1,t_1; r_2, t_2) \]

\[ i \frac{\partial}{\partial t_2} \Psi(r_1,t_1; r_2, t_2) = c \nabla_2 \times \Psi(r_1,t_1; r_2, t_2) \]

one-time wave function:

\[ \Psi(r_1, r_2, t) = \sum_j C_j \psi_j(r_1, t) \otimes \phi_j(r_2, t) \]

one-time Maxwell-Dirac Eqn.

\[ i \frac{\partial}{\partial t} \Psi(r_1, r_2, t) = c \nabla_1 \times \Psi(r_1, r_2, t) + c \nabla_2 \times \Psi(r_1, r_2, t) \]
Obtaining the two-time WvEqns by collapsing the one-time Eqn.

one-time Max-Dirac Eqn.

\[
\frac{i}{c} \frac{\partial}{\partial t} \Psi(r_1, r_2, t) = \nabla_1 \times \Psi(r_1, r_2, t) + \nabla_2 \times \Psi(r_1, r_2, t)
\]

solution:

\[
\Psi(r_1, r_2, t) = \sum_j C_j \psi_j(r_1, t) \otimes \phi_j(r_2, t)
\]

at time \( t = T_1 \), measure \( \vec{r}_1 \), obtain value \( \vec{R}_1 \)

\[
\Psi(r_1, r_2, t) \Rightarrow \Psi(\vec{R}_1, r_2, t) = \sum_j \left[ C_j \psi_j(\vec{R}_1, T_1) \right] \otimes \phi_j(r_2, t)
\]

same form as the two-time wave function before measurement:

\[
\Psi(\vec{r}_1, t_1; \vec{r}_2, t_2) = \sum_j C_j \psi_j(\vec{r}_1, t_1) \otimes \phi_j(\vec{r}_2, t_2)
\]

Obeys the two-time Max-Dirac Eqns.
Two-Photon Disentanglement
Encode qubit values in orbital angular-momentum (OAM) photon states

\[ \Psi_{IN} = \psi_{+1}(r_1) \phi_{-1}(r_2) + \psi_{-1}(r_1) \phi_{+1}(r_2) \]

Assume that all information about the atmospheres is lost - no adaptive optics

\[ \rho = \begin{pmatrix} p_{00} & 0 & 0 & 0 \\ 0 & p_{01} & d & 0 \\ 0 & d^* & p_{10} & 0 \\ 0 & 0 & 0 & p_{11} \end{pmatrix} \]

\[ \text{concurrence} = \max \{ 2 |d| - 2\sqrt{p_{00}p_{11}}, 0 \} \]

Photon

“beam” width

atmosphere scale width-large small

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The “Maxwell-Wolf” Eqns:
In second-order coherence theory, the most general description of a electromagnetic vector field is given by the second-order coherence matrices. [Mandel&Wolf text (almost)]
(tensors)

\[
G(x_1,t_1;x_2,t_2) = \langle E^*(x_1,t_1) \otimes E(x_2,t_2) \rangle \\
H(x_1,t_1;x_2,t_2) = \langle B^*(x_1,t_1) \otimes B(x_2,t_2) \rangle \\
M(x_1,t_1;x_2,t_2) = \langle E^*(x_1,t_1) \otimes B(x_2,t_2) \rangle \\
N(x_1,t_1;x_2,t_2) = \langle B^*(x_1,t_1) \otimes E(x_2,t_2) \rangle
\]

\[
i \frac{\partial}{\partial t_j} J = c \nabla_j \times K, \quad j = 1, 2 \quad (J,K) \in \{(G,iN),(iM,H),(iN,-G),(H,-iM)\}
\]

Recall:

\[
i \frac{\partial}{\partial t_j} \Psi(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2) = c \nabla_j \times \Psi(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2), \quad j = 1, 2
\]
The result generalizes an earlier one by Saleh, Teich, and Sergienko \textit{pri} 94, 223601 (2005).

**“Maxwell-Wolf Eqns”**

\[ i \frac{\partial}{\partial t_j} \mathbf{J} = c \nabla_j \times \mathbf{K}, \quad j = 1, 2 \]

\[(J,K) \in \{(G,iN),(iM,H),(iN,-G),(H,-iM)\}\]

**Wolf Eqns (1954)**

\[
\left( \nabla_j^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t_j^2} \right) \mathbf{F} = 0, \quad j = 1, 2
\]

\[
\mathbf{F} \in \{G,H,N,M\}\]

\[
(i \frac{\partial}{\partial t_j} \bar{\Psi}(\vec{r}_1,t_1;\vec{r}_2,t_2) = \]

\[
c \nabla_j \times \bar{\Psi}(\vec{r}_1,t_1;\vec{r}_2,t_2), \quad j = 1, 2
\]

**two-time Max-Dirac Eqns.**

**two-time, two-photon wv fn**

**two-time Max-Wave Eqns.**
Summary: Field modes vs. Photon states.

• for small number of photons, a wave mechanics with Maxwell-Dirac wave equations is appropriate, but not required.

• EM field wave-packet quantization is closely related to photon space-time states.

• need to be careful with Lorentz invariance and interpretation (no local photon number density, but there is a local photon energy density)

• QST can determine transverse spatial state of an ensemble of photons.

• there are remarkable parallels between classical coherence theory and two-photon wave mechanics. (and differences, A.Z. et. al.)

• two-photon Maxwell-Dirac equation is useful for studying entanglement/disentanglement, etc.
References:


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