In the limiting case \( \delta \gg \lambda \), the amp. of the mag field inside the conductor is indep of frequency.

At low frequency \( H \) indep \( \omega \)

\[
E \omega \quad \Rightarrow \quad Q = \int_0^1 E^2 \omega \, d\omega \propto \omega^2
\]

At high frequency \( H = H_0\) \( e^{-i\omega t} \quad (\omega \rightarrow \infty) \)

\[
E = \frac{\omega}{\sqrt{3\pi \sigma}} (1-i) \hat{H} \times \mu_0 = \frac{\omega}{\sqrt{3\pi \sigma}} (1-i) \hat{H}_0 e^{-i\omega t}
\]

\[
\mathbb{E} = \frac{\omega}{4\pi} \sqrt{\frac{\omega}{3\pi \sigma}} (1-i) \hat{H}_0 \frac{1}{4} e^{-2i\omega t} + e^{2i\omega t} + 2
\]

\[
Q = \frac{c}{16\pi \sqrt{2\pi \sigma}} \int H_0^2 \, df
\]

\( H_0 \) oversurface is given by solving the problem \( H_0 \) field outside a superconductor of the same shape.

High frequencies.
Energy dissipation can also be expressed in terms of the total magnetic moment $\mathbf{M}$ acquired by the conductor into the magnetic field. 

Periodic field $\Rightarrow \mathbf{M}$ periodic 

\[-\mathbf{M} \cdot \frac{d\mathbf{H}}{dt}\] 

rate of change of free energy; $\mathbf{H}$ is uniform external field 

(Does not immediately give the required energy dissipation because of the periodic movement of energy between the body and the surrounding field) 

\[-\mathbf{M} \cdot \frac{d\mathbf{H}}{dt}: \quad \text{(waveforms)}\]

Cycle averaging gives dissipation: 

\[-Q = -\mathbf{M} \cdot \frac{d\mathbf{H}}{dt} \quad \frac{d\mathbf{H}}{dt} = -i\omega \mathbf{H}\]

\[\mathbf{M} = m_R + im_L\]

\[\mathbf{H} = h_R + ih_L\quad \dot{\mathbf{H}} = -i\omega h_R + \omega h_L\]

\[Q = -\omega m_L h_L = -\frac{i}{4}(\mathbf{M} + \mathbf{M}^*)(\mathbf{H} + \mathbf{H}^*)\]

\[= -\frac{i}{4}(\mathbf{M} \cdot \mathbf{H}^* + \mathbf{M}^* \cdot \mathbf{H}) = -\frac{i}{2} \text{Re}(\mathbf{M} \cdot \mathbf{H}^*)\]

\[Q = -\frac{i}{2} \text{Re}(i\omega \mathbf{M} \cdot \mathbf{H}^*) = \frac{\omega}{2} \text{Im}(\mathbf{M} \cdot \mathbf{H}^*)\]
\[ M_i = \text{Vol} \cdot k_i \cdot h_i \]

\( k_i \) depend on shape of the body and its orientation into external field. BUT not its volume.

\( \text{Vol} \) magnetic polarizability tensor of the body as a whole (a generalized susceptibility).

\[ M \cdot h_i^* = \text{Vol} \cdot k_i \cdot h_i \cdot h_i^* = \frac{1}{2} \text{Vol} \cdot (h_i^* \cdot h_i + h_i \cdot h_i^*) = \]

\[ \text{using symmetry of } \text{Vol} \]

\[ \to = \text{Vol} \cdot k_i \cdot \text{Re} \left( h_i \cdot h_i^* \right) \]

\[ \alpha_i^* = \alpha_i^* + i \alpha_i^* \]

\[ Q = \frac{\omega}{2} \text{Im} \left( M \cdot h_i^* \right) = \frac{\omega}{2} \alpha_i^* \text{Re} \left( h_i \cdot h_i^* \right) \]

---

**Energy Dissipation**

- Determined by the imaginary part of magnetic polarizability.
- At low frequency \( \omega < \omega_c \) \( \alpha_i^* \propto \omega \)
- At high frequency \( \omega > \omega_c \) \( \alpha_i^* \propto 1/\omega \)

**Magnetic Moment**

- In a variable magnetic field, the field is mostly due to demagnetizing currents setup in the body. Not zero even if \( \mu = 1 \), when the magnetic moment in static field vanishes.

\[ \lim_{w \to 0} M(w) = \lim_{w \to 0} \text{Vol} \cdot k_i \cdot \frac{1}{w} \cdot h_i \]

\( w \to \infty \) magnetic field cannot penetrate the body. \( \alpha_i^* \) tends to a different constant limit, corresponding to the static magnetic moment of a superconductor of the same shape.

\[ \lim_{w \to 0} \text{Vol} \cdot k_i \cdot \frac{1}{w} \cdot h_i \to \text{static} \]

\( w \to \infty \) tends to constant limit (which must be zero if \( \mu = 1 \)).
Problem #1 § 59 (Depth of Penetration of a Magnetic Field into a Conductor)

Determine the magnetic polarizability of an isotropic conducting sphere with radius $a$ in a uniform periodic external field.

The field inside the sphere obeys $\nabla^2 \mathbf{H}^{(i)} + k^2 \mathbf{H}^{(i)} = 0$ where

$\nabla \cdot \mathbf{H}^{(i)} = 0 \quad k = \frac{1+i}{\delta}$

we write $\mathbf{H}^{(i)} = \nabla \times \mathbf{A}$ (since $\nabla \cdot (\nabla \times \mathbf{A})$)

$0 = \nabla^2 (\nabla \times \mathbf{A}) + k^2 (\nabla \times \mathbf{A})$

$\nabla \times \{ \nabla^2 \mathbf{A} + k^2 \mathbf{A} \} = 0$

$(\nabla^2 + k^2) \mathbf{A} = \nabla \mathbf{g}$

should be enough (?) but $L \neq L$

By symmetry, the only constant vector on which the required solution can depend is the external field $\mathbf{h}$.

$\mathbf{F} = \frac{1}{r} \sin hr$

(spherically symmetric solution $(\nabla^2 + k^2) \mathbf{F} = 0$

finite at the origin)

$(\nabla^2 + k^2) \mathbf{F} = \left( \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + k^2 \right) \frac{1}{r} \sin hr$

$= \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \left( -\frac{1}{r^2} \sin hr + \frac{k}{r} \cos hr \right)$

$+ \frac{k^2}{r} \sin hr$

$\delta = \epsilon / \sqrt{2 \pi \mu_0}$
\[ (\mathbf{A} + 2\mathbf{r}) \cdot \mathbf{F} = \frac{1}{r^2}\left( - \frac{1}{2} \mathbf{r} \cdot \mathbf{A} + 2\mathbf{r} \cdot \mathbf{F} \right) + \mathbf{k}^2 \mathbf{F} \mathbf{r} \]

Then the polar vector \( \mathbf{A} \), which satisfies the vector eq. \( \nabla^2 \mathbf{A} = 0 \) and is supposed linearly on the earth vector \( \mathbf{r} \) can be written as

\[ \mathbf{A} = \mathbf{F} \cdot \mathbf{r} \mathbf{A} \mathbf{F} \mathbf{r} \]

Here

\[ \nabla^2 \mathbf{F} = 0 \]

\[ \mathbf{F} \mathbf{r} = \mathbf{F} \cdot \mathbf{r} \mathbf{F} \]
\[ H^{(i)} = \beta \nabla (\nabla \cdot f \hat{h}) - \beta \nabla^2 (f \hat{h}) \]

\[ \beta \nabla (\nabla \cdot (f \hat{h})) = \beta \nabla (\frac{\hat{h} \cdot \nabla f}{r}) = \beta \nabla (\frac{\hat{h} \cdot \nabla f}{r^2}) = \beta \nabla (\hat{h} \frac{\partial f}{\partial r} - \frac{\partial f}{\partial z}) \]

\[ r = (x^2 + y^2 + z^2)^{1/2} \]

\[ \frac{\partial r}{\partial z} = \frac{\hat{z} \cdot \hat{z}}{r^2} = \frac{\hat{z} \cdot \hat{z}}{r} = \cos \theta \]

\[ = \beta \hat{h} \cdot \nabla \left( \frac{\partial f}{\partial r} \cos \theta \right) \]

\[ = \beta \hat{h} \cdot \left( \frac{n \partial f}{\partial r} + \frac{\hat{\theta} \cdot \hat{z}}{r} \frac{\partial f}{\partial \theta} \right) \frac{\partial f}{\partial r} \cos \theta \]

\[ = \beta \hat{h} \left( \frac{n}{\partial f} \cos \theta - \frac{\hat{\theta} \cdot \hat{z}}{r} \frac{\partial f}{\partial \sin \theta} \right) \]

\[ \hat{h} \hat{\theta} \hat{\omega} \cdot \left( \hat{h} \cos \theta \right) \hat{n} = (\hat{h} \cdot \hat{n}) \hat{n} \]

\[ n \hat{h} \cos \theta - \hat{\theta} \hat{n} \sin \theta = \hat{n} \Rightarrow -\hat{\theta} \hat{n} \sin \theta = \hat{n} - (\hat{n} \cdot \hat{n}) \hat{n} \]

\[ = \beta (\hat{n} \cdot \hat{n}) \hat{n} + \beta (\hat{n} - (\hat{n} \cdot \hat{n}) \hat{n}) \hat{n} \]
Let's get rid of $\xi$:

\[-k^2\xi = \nabla^2 \xi = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial \xi}{\partial r}) = \frac{1}{r^2} \left( 2r \frac{\partial \xi}{\partial r} + r^2 \frac{\partial^2 \xi}{\partial r^2} \right) = \frac{2}{r} \xi' + \xi'' \]

\[\therefore \xi'' = -k^2\xi - \frac{2}{r} \xi'\]

\[\beta' \nabla (\nabla \cdot (n \phi)) = \beta' (n \cdot n) \nabla (\frac{k^2\xi}{r} - \frac{2}{r} \xi') + \beta \frac{n \cdot \xi'}{r} - \beta \frac{n \cdot n}{r} \frac{\xi'}{r}
\]

\[= -\beta \frac{n \cdot n}{r} \left( \frac{k^2\xi}{r} + \frac{2}{r} \xi' \right) + \beta \frac{n \cdot \xi'}{r}
\]

Substitution at the top of page 3 gives:

\[H^{(e)} = \beta \frac{n}{r} \left( \frac{\xi'}{r} + k^2\xi \right) + \beta \frac{n (n \cdot n)}{r} \left( \frac{k^2\xi}{r} + \frac{2}{r} \xi' \right)\]

\[\hat{H}^{(e)} \quad \text{field outside:} \quad \nabla \times \hat{H}^{(e)} = 0 \]

\[\nabla \cdot \hat{H}^{(e)} = 0 \]

We put:

\[H^{(e)} = -\nabla \phi + \mathbf{n} \quad \nabla \times H^{(e)} \quad \text{satisfied automatically}
\]

\[0 = \nabla \cdot H^{(e)} = -\nabla^2 \phi + \mathbf{n} \cdot \frac{\xi'}{r} \implies \nabla^2 \phi = 0
\]

$\phi$ depends linearly on the constant $\mathbf{n}$.

$\phi$ vanishes at infinity.

$\mathbf{H}^{(e)} = -\nabla \phi \cdot \frac{1}{r}$

This function obeys Laplace's eqn away from $\mathbf{n}$.
\[ H^{(e)} = V \alpha \nabla \left[ \mathbf{r} \cdot \nabla \left( \frac{1}{r} \right) \right] + \mathbf{h} \]

\[ \frac{2}{3x} \frac{1}{r} = -\frac{1}{2} \frac{2x}{r^3} = -\frac{x}{r^3} \]

\[ \nabla \frac{1}{r} = -\frac{x}{r^3} \quad \nabla \nabla \frac{1}{r} = -\nabla \frac{1}{r^3} = -\left( \nabla \frac{1}{r^3} \right) \frac{x}{r^3} \]

\[ \nabla \mathbf{c} = \left( \frac{\hat{x}}{\partial x} + \frac{\hat{y}}{\partial y} + \frac{\hat{z}}{\partial z} \right) \left( \hat{x} + \frac{1}{r} \hat{y} + \frac{1}{r} \hat{z} \right) \]

\[ \nabla \frac{1}{r^3} = \nabla \left( \frac{1}{r^3} \right) = 3 \left( \frac{1}{r^4} \right) \nabla \frac{1}{r} = -3 \frac{x}{r^5} \]

\[ \nabla \nabla \frac{1}{r} = 3 \frac{r^2}{r^5} - \frac{x}{r^3} \]

\[ \nabla \nabla \frac{1}{r} = \frac{1}{r^3} \left[ 3 \mathbf{n} \mathbf{n} - \frac{1}{r^2} \right] \]

\[ H^{(e)} = \frac{V \alpha}{r^3} \left[ 3(\mathbf{n} \cdot \mathbf{n}) \mathbf{n} - \mathbf{h} \right] + \mathbf{h} \]

Field from a dipole \( \mathbf{m} \) is

\[ \frac{\mathbf{m}}{r^3} \cdot \left[ 3 \mathbf{n} \mathbf{n} - \frac{1}{r^2} \right] \]

so we conclude that

\( V \alpha \mathbf{h} \) is the dipole moment, \( \mathbf{n} \) is a unit vector, \( \mathbf{h} \) is the sphere, and \( V \alpha \mathbf{h} \) is its magnetic polarizability.
Problem 1 § 59 continued

on the surface of the sphere \( r = a \). The components of \( \mathbf{H} \) must be continuous (Eq. (58.8) is in force because \( \mu = 1 \) inside the conducting sphere).

Normal component:

\[
\mathbf{H}^{(e)} \cdot \mathbf{n} = \mathbf{H}^{(e)} \cdot \mathbf{n}
\]

\[
\beta \left( \frac{F'(a)}{a} + k^2 \chi(a) \right) (\mathbf{n} \cdot \mathbf{n}) - \beta \left( \frac{\delta(a)}{a} + k^2 \chi(a) \right) (\mathbf{n} \cdot \mathbf{n})
\]

\[
= \frac{4\pi}{3} \alpha \left[ 3(\mathbf{n} \cdot \mathbf{n}) - \mathbf{n} \cdot \mathbf{n} \right] + \mathbf{n} \cdot \mathbf{n}
\]

\[
- \beta \left( \frac{2F'}{a} \right) = \frac{8\pi}{3} \alpha + 1
\]

Transverse component:

\[
\mathbf{H}^{(t)} = (\mathbf{H}^{(e)} \cdot \mathbf{n}) \mathbf{n} = \mathbf{H}^{(e)} - (\mathbf{H}^{(e)} \cdot \mathbf{n}) \mathbf{n}
\]

\[
\beta \left( \frac{F'}{a} + k^2 \chi \right) (\mathbf{n} - (\mathbf{n} \cdot \mathbf{n})) = \left[ -\frac{4\pi}{3} \alpha + 1 \right] (\mathbf{n} - (\mathbf{n} \cdot \mathbf{n}))
\]

\[
\beta \left( \frac{F'}{a} + k^2 \chi \right) = -\frac{4\pi}{3} \alpha + 1
\]

\[
- \left( \frac{8\pi}{3} \alpha + 1 \right) \left( \frac{a}{2F'} \right) \left( \frac{F'}{a} + k^2 \chi \right) = -\frac{4\pi}{3} \alpha + 1
\]

\[
\left( \frac{8\pi}{3} \alpha + 1 \right) \left( \frac{1}{2F'} + \frac{a k^2 \chi}{2F'} \right) = \frac{4\pi}{3} \alpha - 1
\]
\[
\frac{4\pi^2}{3} \frac{ak^2 f}{f'} + \frac{1}{2} + \frac{ak^2 f}{2f'} = -1
\]

\[
\frac{4\pi^2}{3} \frac{ak^2 f}{f'} = -\frac{3}{2} - \frac{ak^2 f}{2f'}
\]

\[
\frac{4\pi^2}{3} = -\frac{3}{2} \frac{f'}{ak^2 f} - \frac{1}{2}
\]

\[
\chi = -\frac{3}{8\pi} - \frac{9}{8\pi} \left( \frac{1}{2a} \cot ka - \frac{1}{12a^2} \right)
\]

\[
f = \frac{1}{r} \sin kr
\]

\[
f' = \frac{kr}{r^2} \cos kr - \frac{1}{r^2} \sin kr
\]

\[
\frac{f'(a)}{f(a)} = k \cot ka - \frac{1}{a}
\]
Problem 2 §5.89 conducting cylinder (with radius $a$) in a uniform periodic magnetic field perpendicular to its axis.

2-D analogue of all vector operations in a plane perpendicular to the axis of the cylinder, and $z$ is a position in that plane.

$f$ is cylindrically symmetric solution finite for $r = 0$

$$\nabla^2 \left( x H_x^{(i)} + i y H_y^{(i)} \right) + k^2 H^{(i)} = 0$$

$$\nabla^2 + k^2 \left( x \frac{\partial^2 A}{\partial y^2} - y \frac{\partial^2 A}{\partial x^2} \right) = 0$$

Take

$$\nabla^2 + k^2 \Delta A = 0$$

$H$ is an axial vector (doesn't change under inversion of spherical coordinates)

$A$ is polar

only (spatially) constant vector on which solen can depend is $N$. $f = \phi(r)$ cylindrically symmetric solution of

$$\nabla^2 + k^2 \Delta f = 0$$

finite for $r = 0$. $f$
Bessel Functions

Butkov "Mathematics Physics" section 9.7

Bessel DE
\[
\frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \left(1 - \frac{m^2}{x^2}\right)y = 0
\]
\[
\frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \left(1 - \frac{\mu^2}{x^2}\right)y = 0
\]
\(m\) often an integer, but sometimes not
\(\mu^2\) non-negative

Frobenius series
\[
y(x) = \sum_{n=0}^{\infty} a_n x^{s+n}
\]
\[
x^2 y'' + xy' + (x^2 - \mu^2)y = 0
\]

\[
\sum_{n=0}^{\infty} \left[ a_n (s+n)(s+n-1) + a_{n-1} (s+n) + x^2 a_{n-2} - \mu^2 a_n \right] x^{s+n} = 0
\]
\[
\sum_{n=0}^{\infty} a_n \left( (s+n)^2 - \mu^2 \right) x^{s+n} + \sum_{n=0}^{\infty} a_n x^{s+n+2} = 0
\]
\[
\sum_{n=0}^{\infty} a_n \left( (s+n)^2 - \mu^2 \right) x^{s+n} + \sum_{n=2}^{\infty} a_{n-2} x^{s+n} = 0
\]

\[a_0 \left[ s^2 - \mu^2 \right] = 0 \quad \text{indicial equation} \quad \Rightarrow \quad s = \pm \mu \]

\[-a_n \left( (s+n)^2 - \mu^2 \right) = a_{n-2}\]

\[a_n = -\left( (s+n)^2 - \mu^2 \right)^{-1} a_{n-2} \quad \text{for } s = \mu, \text{ this becomes}\]

\[a_n = \left( 2\mu^2 + n^2 \right)^{-1} a_{n-2}\]

\[
a_n = \frac{a_{n-2}}{n \left[ 2 \mu + n \right]} \quad \text{for } n \geq 2
\]

all even coeffs follow from \(a_0\)
We also have

\[ a_1 \left[ (s+1)^2 - u^2 \right] = a_1 \left[ 2u + 1 \right] = 0 \Rightarrow a_1 = 0 \]

\[ \Rightarrow \text{all odd coeffs vanish.} \]

\[ y(x) = x^u \sum_{n=0}^{\infty} a_n x^n = x^u \text{ times an even function of } x \]

Let \( n = 2k \)

\[ a_{2k} = -\frac{a_{2k-2}}{2k \left[ 2u + 2k \right]} = \frac{a_{2k-2}}{4k (u+k)} \quad k \geq 1 \]

\[ a_{2k} = \frac{1}{4^k k!} \frac{(-1)^k}{(u+k) \cdots (u+2)(u+1)} a_0 \]

standardize solution by choosing \( a_0 = \frac{1}{2^u \Gamma(1+u)} \)

\[ a_{2k} = \frac{(-1)^k}{2^{2k} + u} \frac{1}{k! \Gamma(1+u) \Gamma(u+k+2) \cdots \Gamma(u+k) / \Gamma(u+3)} \]

\[ \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \]

\[ J_u(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k! \Gamma(u+k+1) 2^{u+2k} x^{u+2k}} \]

Bessel function of order \( u \) (of first kind)
ratio of successive terms

\[
\frac{x^2}{(k+1) \Gamma (u+k+1)} \frac{1}{(u+k+2)} Z^2
\]

\[
= - \frac{x^2}{(k+1)(u+k+1)} Z^2 \quad \rightarrow \quad 0 \quad \text{as} \quad k \rightarrow \infty
\]

If \( \mu = m \) is an integer, then \( J_m(x) \) is single valued, and the above series is a Maclaurin series.

\( \mu \) is not an integer \( \Rightarrow \) branch pt at origin.

solutions with \( s = -\mu \)

\[ a_1, [ (1-\mu)^2 - \mu^2 ] = 0 \quad a_1 \left[ 1 - 2\mu \right] = a_1 \]

\[ a_n = - \frac{1}{[ (n-\mu)^2 - \mu^2 ]} \quad a_{n-2} = - \frac{1}{n \left[ n-2\mu \right]} a_{n-2} \]

difficulties when

(a) \( \mu \) is an integer

(b) \( \mu \) is a half-integer

neither (a) nor (b) \( \Rightarrow \) \( a_1 = 0 \) along w all odd coeffs

Bessel functions of the first kind of order \( -\mu \)

\[
J_{-\mu}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k! \Gamma (k-\mu+1)} x^{2k-\mu}
\]

(linearly indep of \( J_\mu(x) \))

general soln of Bessel DE: \( c_1 J_\mu(x) + c_2 J_{-\mu}(x) \quad \mu \neq \text{half-integer} \)
\[ u \text{ half integer } \Rightarrow 2u = \text{odd integer} \]
\[ a_n = \frac{1}{2^{n(n-2u)}} a_{n-2} \]
defines all even coeffs in terms of \( a_0, a_2 \) before.

The recurrence formula breaks down for odd coeffs:
\[ a = a_1, a_3, \ldots, a_{2u-2} \text{ but } a_{2u}, a_{2u+2} \text{ not necessarily} \]

But there is nothing to prevent us from setting \( a_{2u} = 0 \) by our own choice (all subsequent odd coeffs must be zero) and we can define \( J_{-u}(x) \) exactly as before.

\( J_{-u} \) remains a continuous function of \( u \) as \( u \) passes through a half-integer value.

\[ y(x) = c_1 J_u(x) + c_2 J_{-u}(x) \]
with \( u \) half-integer.

\[ a_0 = \frac{1}{2^{-u} \Gamma(1-u)} \]
\[ \Gamma(1-u) \text{ is } \infty \text{ when even } u \text{ is a positive integer} \]

\[ \text{eg. } \Gamma(0) = \int_0^\infty \frac{e^{-t}}{t} dt \]

\[ a_{2k} = (-1)^k \frac{1}{\Gamma(-u+k+1)} 2^{2k-u} \]

providing we adopt the logical convention that \( a_{2k} = 0 \) for \( u-k > 0 \) (because \( \Gamma(-u+k+1) \) has a pole).

Summation effectively starts at \( k = u \) (we set \( u = m \) in accordance with our convention abt Greek \& Latin letters).

\[ J_m(x) = \sum_{k=m}^{\infty} (-1)^k \frac{1}{k! \Gamma(k-m+1)} 2^{2k-m} x^{-m+2k} \]

\[ k' = k-m \]

\[ \sum_{k'=0}^{\infty} (-1)^{k'} \frac{x^{2k'+m}}{(k'+m)! \Gamma(k'+1)} 2^{k'+m} \]
\[(\frac{k}{m} + m)! \Gamma(\frac{k}{m} + 1) = (\frac{k}{m})! \prod \frac{1}{(\frac{k}{m} + 1)(\frac{k}{m} + 2)} \ldots (\frac{k}{m} + m) \]
\[= \frac{k}{m}! \Gamma(\frac{k}{m} + m + 1) \]

This is nothing else but \((-1)^m J_m(x)\)
we've lost linear independence of the solutions \(J_u + J_{-u}\).
when the roots of the indicial equation differ by an integer, one of the solutions
of the Bessel DE may have a logarithmic singularity
second order may be sought and obtained in the form
\[y_2(x) = \log x \sum_{n=0}^{\infty} c_n x^{n+\nu} + \sum_{\nu=0}^{\infty} a_\nu x^{\nu+p} \]

Depending on standardization, known as Bessel functions 2nd kind: \(Y_m(x)\)
or Neumann functions \(N_m(x)\).

From the Frobenius series for \(J_0(x)\) it follows that
\[J_0(x) = x^\nu F(x) \]

\(u(x)\) is an analytic and nonvanishing at origin

\[J_0(0) = 0 \quad \text{unless} \quad \nu = 0 \quad J_0(0) = \frac{1}{\Gamma(1)} = \frac{1}{\int_0^\infty e^{-t} \, dt} \]

In general
\[J_0(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k! \Gamma(k+1)} x^{2k} \]

\[\Gamma(k+1) = \Gamma(k) = \int_0^\infty dt \, t^k e^{-t} \]

\[v = t^k \quad dv = e^{-t} \, dt \quad du = h^k \, dt \quad v = -e^{-t} \]

\[\Gamma(k+1) = -t^k e^{-t} \left[ \int_0^\infty dt \, t^k e^{-t} \right] \]

\[= k \Gamma(k) \quad \Gamma(0) = \frac{1}{\Gamma(1)} \]

\[J_0(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k! k!} \frac{1}{2^{2k}} x^{2k} \]

\[= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!!} \left[ \frac{x^{2k}}{(2k)!} \right] \]
\[ f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k! \Gamma(m+k+1)} \frac{1}{2^{k+m}} x^{2k+m} \]

\[ f'(x) = \left( \frac{J_m(x)}{x^m} \right)' = \sum_{k=1}^{\infty} (-1)^k \frac{2k}{k! \Gamma(m+k+1)} \frac{1}{2^{k+m}} x^{2k-1} \]

\[ = \sum_{k=1}^{\infty} (-1)^k \frac{1}{(k-1)! \Gamma(m+k+1)} \frac{1}{2^{k+m-1}} x^{2k-1} \]

\[ = -\sum_{k=0}^{\infty} (-1)^k \frac{1}{k! \Gamma(m+k+2)} \frac{1}{2^{k+m+1}} x^{2k+1} \]

\[ = -x \sum_{k=0}^{\infty} (-1)^k \frac{1}{k! \Gamma(m+1+k+1)} \frac{1}{2^{k+m+1}} x^{2k} \]

\[ = - \frac{J_{m+1}(x)}{x^m} \]

\[ \frac{d}{dx} \left[ \frac{J_m(x)}{x^m} \right] = f'(x) = \frac{J_{m+1}(x)}{x^m} \quad m \geq 0 \]

\[ \frac{J_m}{x^{m+1}} + \frac{1}{x^m} J'_m = -\frac{J_{m+1}}{x^m} \]

\[ J'_m = -J_{m+1} + \frac{m}{x} J_m \]

\[ \mu x^{\mu-1} J_\mu + x^\mu \frac{dJ_\mu}{dx} = x^\mu J_{\mu-1} \]

\[ J'_\mu = J_{\mu-1} - \frac{\mu}{x} J_\mu \]

\[ J_{\mu+1} + J_{\mu-1} = \frac{\mu}{x} J_\mu \quad \text{OK} \]

\[ \frac{dJ_\mu}{dx} = J_{\mu-1} - J_{\mu+1} \quad \text{OK} \]
For Bessel functions of integral order there also exists a generating function

\[ e^{(t - \frac{1}{4}t^2)/2} = \sum_{m=-\infty}^{\infty} t^m J_m(t) \quad t \geq 0 \]

\[ t = e^{i\theta} \quad e^{i\theta} - e^{-i\theta} = 2i \sin \theta \]

\[ e^{ix \sin \theta} = \sum_{m=-\infty}^{\infty} e^{im\theta} J_m(x) \quad \text{Fourier Series for} \quad e^{ix \sin \theta} \]

\[ g = R \otimes Z \]

\[ \frac{1}{r^2} \frac{d}{dr} \left( r \frac{dR}{dr} \right) + \frac{1}{r^2} \frac{d^2\theta}{d\theta^2} + \frac{1}{r^2} \frac{d^2z}{dz^2} + k_z^2 = 0 \]

\[ \frac{1}{r} \frac{d}{dr} \left( r \frac{dR}{dr} \right) + \frac{1}{r^2} \frac{d^2\theta}{d\theta^2} + k_z^2 r^2 = 0 \]

\[ \lambda_z \]

\[ \frac{d}{dr} \left( r \frac{dR}{dr} \right) + \left( \frac{\lambda_z}{r} + k_z^2 r \right) R = 0 \]

In most cases \( \lambda_z = -m^2 \)

\[ \frac{d}{dr} \left( r \frac{dR}{dr} \right) + \left( k_z^2 r - \frac{m^2}{r} \right) R = 0 \]

w/ BC on \( R \)

becomes an eqn for \( k_z^2 \)

at most commonly types of physical conditions

\[ r = 0 \quad \text{(a) } R(r) \text{ must be finite} \]

\[ r = a \quad \text{(b) at } R(r) \text{ must satisfy } R(2a) = 0 \text{ (Dirichlet)} \]

\[ AR + B \frac{dR}{dr} = 0 \text{ (Intermediate) } \]

\[ \frac{dR}{dr} \bigg|_{r=a} = 0 \text{ (Neuman)} \]
or orthogonality:

\[ \frac{d}{dr} \left( r \frac{dR_1}{dr} \right) - \frac{m^2}{r} R_1 = -k_1^2 r R_1 \]

\[ \int_0^a \left\{ \frac{d}{dr} \left[ r \frac{dR_1}{dr} \right] - R_1 \frac{d}{dr} \left[ r \frac{dR_2}{dr} \right] \right\} dr = (k_2^2 - k_1^2) \int_0^a r R_1 R_2 \]

\[ \left( R_2 \frac{dR_1}{dr} - R_1 \frac{dR_2}{dr} \right)_0^a - \int_0^a r \frac{d}{dr} \left[ \frac{dR_2}{dr} \frac{dR_1}{dr} - \frac{dR_1}{dr} \frac{dR_2}{dr} \right] \]

\[ a = \int_0^a r \, R_1 R_2 \quad \text{if} \quad k_1^2 \neq k_2^2 \]

\[ n = 1, 2, 3, \ldots \quad \text{nontrivial roots of } J_n(x) \]

values of \( k \) for Dirichlet conditions: \( k_{mn} = \alpha_{mn}/a \)

\[ \beta_{mn} = 1, 2, \ldots \quad \text{roots of } J'_n(x) \]

useful for Neuman condition

infinite set of mutually orthogonal functions. \( R_k(r) = J_m (k m n r) \)

for any reasonably well-behaved function \( f(r) = \sum_{n=1}^{\infty} a_{mn} J_m(k m n r) \)

\[ a_n = a \int_0^a r J_m(k m n r) f(r) \]

\[ \int_0^a \left| J_m(k m n r) \right|^2 \]
\[
\frac{d}{dr} \left( r \frac{dR}{dr} \right) - \frac{m^2}{r} R = -k^2 r R
\]

\[
\frac{d}{dr} \left( r \frac{dJ_m(kr)}{dr} \right) - \frac{m^2}{r} J_m(kr) = -k^2 r J_m(kr)
\]

\[
\left\{ J_m(kr) \frac{d}{dr} \left( r \frac{dJ_m(kr)}{dr} \right) - J_m(kr) \frac{d}{dr} \left( r \frac{dJ_m(kr)}{dr} \right) \right\}
\]

\[
= (k^2 - k_{mn}^2) \int_0^a dr r J_m(kr) J_m(k_{mn}r)
\]

\[
L_0 = \left\{ r J_m(kr) \frac{dJ_m(k_{mn}r)}{dr} - r J_m(k_{mn}r) \frac{dJ_m(kr)}{dr} \right\}
\]

\[
= a \left\{ J_m(ka) \frac{dJ_m(k_{mn}a)}{dr} - J_m(k_{mn}a) \frac{dJ_m(ka)}{dr} \right\}
\]

assuming Dirichlet

and valid for all \( a \) whenever

\[
a J_m(ka) k_{mn} J_m(k_{mn}a) = (k^2 - k_{mn}^2) \int_0^a dr r J_m(ka) J_m(k_{mn}r)
\]

\[
a a J_m(ka) k_{mn} J_m(k_{mn}a) = 2k \int_0^a dr r J_m(ka) J_m(k_{mn}r)
\]

\[
+ (k^2 - k_{mn}^2) \int_0^a dr r^2 J_m(ka) J_m(k_{mn}r)
\]

\[
\text{diff' wrt } k \text{ yields:}
\]
now let $k \to k \nu n$.

$$\frac{a^2}{2} \left[ J_m \left( kr \nu n a \right) \right]^2 = \int_0^a r \, dr \, \left[ J_m \left( kr \nu n n \right) \right]^2 \equiv N_k$$

Bessel functions do not usually list their derivatives. The derivatives of $J_m (x)$.

$p. 259$:

$$J_{n+1} = \frac{2}{x} J_n - J_{n-1}$$

$$\frac{2}{x} J_n - J_{n-1} - J_{n+1} = -2 J'_n$$

$$\frac{d}{dx} J_n = J_{n-1} - \frac{n}{x} J_n$$

If $x$ is a root of $J_n (x)$, we have

$$\frac{d J_n (x)}{dx} = J_{n-1} (x) = - J_{n+1} (x)$$

$$\int_0^a r \, dr \, \left[ J_m \left( kr \nu n n \right) \right]^2 = \frac{a^2}{2} \left[ J_{m+1} \left( kr \nu n n a \right) \right]^2$$

---

Free vibrations of a circular membrane

- **Cylindrical Membrane**

  \( \frac{\partial^2 u}{\partial t^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \) \quad \text{Poisson's equation}

  \( u(r, \theta, t) = 0 \) \quad \text{Dirichlet} \quad \text{Boundary conditions}

  \( u(r, \theta, t) = u_0 (r, \theta) \) \quad \text{Initial condition}

\( u = R(\theta) T(t) \) \quad \text{Separation of variables}

\( \frac{d}{dt} \) \quad \text{Temperature}

\( \frac{d^2 R}{d\theta^2} = \lambda R \)

\( \frac{1}{r} \frac{d}{dr} \left( r \frac{dR}{dr} \right) + \left( \frac{\lambda}{r^2} - \lambda \right) R = 0 \)

\( \lambda = \lambda_n \) \quad \text{Eigenvalues}

\( T = \frac{c^2}{\mu} \) \quad \text{Mass moment}
\[\lambda \leq 0 \quad \lambda = -k^2 \quad k^2 c^2 = \omega^2 \quad \text{w may be either positive or negative}
\]

\[T(t) = e^{-i\omega t}
\]

\[\lambda_1 = -m^2 \quad \Theta = e^{-i m \theta}
\]

\[\text{for periodicity } m = 0, 1, 2 \ldots
\]

\[\Theta_m(\theta) = A_m \cos m \theta + B_m \sin m \theta
\]

\[\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \left( k^2 - \frac{m^2}{r^2} \right) R = 0
\]

general solution

\[R(r) = c_j J_m(k r) + c_2 N_m(k r)
\]

\[\zeta = 0 \quad J_m(k a) = 0 \quad \text{Dirichlet}
\]

\[k_{mn} = \alpha m n / a
\]

\[\text{entire solution} = \text{double series}
\]

\[u(r, \theta, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\alpha m a / r) \left[ (A_{mn} e^{i \omega m n t} + A_{mn}^* e^{-i \omega m n t}) \cos m \theta + (B_{mn} e^{i \omega m n t} + B_{mn}^* e^{-i \omega m n t}) \sin m \theta \right]
\]

\[A_{mn} + B_{mn}^* \text{ follow from initial conditions}
\]

\[u_0(r, \theta) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\alpha m a / r) \left[ (A_{mn} + A_{mn}^*) \cos m \theta + (B_{mn} + B_{mn}^*) \sin m \theta \right]
\]

\[\int_0^{2\pi} \int_0^a \left[ J_m'(\alpha m'n / a) \right] \cos m' \theta u_0(r, \theta) = \pi \sum_{n=0}^{a} \int_0^{\pi} \int_0^a J_m(n / \alpha) \left[ J_{m+1}(\alpha m'n / a) \right] \cos m \theta \]

\[\times J_{m'n}(\alpha m'n / a) \left[ (A_{m'n} + A_{m'n}^*) \right] = \frac{\pi a^2}{2} \left[ J_{m+1,n}(\alpha m'n / a) \right]
\]
long steel cylinder, radius \( b = 10 \text{cm} \)
constant temp 100\(^{\circ}\text{C} \) initially uniform temp 500\(^{\circ}\text{C} \)
temp at center? \( q \) at \( t = 4 \text{min} \)
rate of heat loss? \( S \)
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) = \frac{1}{a^2} \frac{\partial u}{\partial t}
\]
boundary condition
\( u(b,t) = u_1 = 100^{\circ}\text{C} \)
initial condition
\( u(r,0) = u_0 = 500^{\circ}\text{C} \)
\[
a^2 = k/dp = 0.126 \text{cm}^2/\text{sec}
\]
\( u(r,t) = R(r)T(t) \)
\[
\frac{d}{dr} \left( r \frac{dR}{dr} \right)
\]
conducting cylinder of radius \( a \) with magnetic field parallel to the axis of the cylinder.

By symmetry, the magnetic field is everywhere parallel to the axis of the cylinder.

Outside the cylinder, \( \nabla \times \mathbf{H} = 0 \)

\[
\int_{\text{cylinder}} \nabla \times \mathbf{H} = \int \mathbf{H} \cdot \nabla \mathbf{e}_z = 0
\]

Inside the cylinder, the magnetic field is everywhere proportional (and parallel) to \( \mathbf{H} \):

\[ \mathbf{H} \propto \mathbf{e}_z \]

From Eq. (59.1) \( \nabla^2 \mathbf{H} = -k^2 \mathbf{H} \)

Since \( \mathbf{H} \) is constant, substitution leads to the requirement that

\[ \nabla^2 f + k^2 f = 0 \]

and the cylindrically symmetric solution is required.
We want the Bessel function solution \( J_0(kr) \) rather than the Neumann function solution \( N_0(kr) \), which diverges logarithmically at the origin.

\[
H^{(i)} = \frac{\partial}{\partial r} \frac{J_0(2r)}{J_0(ka)}
\]

de nominator

The current is azimuthal.

\[
\frac{4\pi}{c} \int \varphi = (\text{curl } H^{(i)}) \cdot \mathbf{B} = \frac{\partial H^{(i)}}{\partial t} - \frac{\partial H^{(i)}}{\partial t}.
\]

\[
= -\frac{\partial H^{(i)}}{\partial r} = -\frac{h}{4\pi} \frac{\partial J_0(2r)}{\partial r}
\]

\[
\mathbf{j} = -\frac{ch}{4\pi} \frac{1}{J_0(ka)} \frac{\partial J_0(2r)}{\partial r}
\]

Magnetic moment per unit length, \( M \), is

\[
M = \pi a^2 \int j \cdot B
\]

\[
eq \frac{1}{2\pi} \int \mathbf{d}s \times \mathbf{H}(r) \quad \text{(see 3.1)}
\]

\[
eq -\frac{\mu_0 h}{4\pi} \frac{1}{J_0(ka)} \frac{1}{2} \left( \int_0^a r dr \frac{\partial}{\partial r} + \frac{\partial J_0(2r)}{\partial r} \right)
\]

\[
= -\frac{\mu_0 h}{4} \frac{1}{J_0(ka)} \int_0^a r dr a^2 \frac{\partial J_0(2r)}{\partial r}
\]
\[
\int_0^\infty dx x^2 \frac{\partial J_0 (x)}{\partial x} = \frac{1}{k^2} \int_0^\infty dx x^2 \frac{dJ_0 (x)}{dx} \\
= \frac{2a}{k} J_0 (ka) \quad \text{Integration by parts:} \quad x^2 \frac{dJ_0 (x)}{dx} = \frac{d}{dx} [x^2 J_0] - 2x J_0 \\
= a^2 J_0 (ka) - \frac{2a}{k} \int_0^\infty dx x J_0 (x) \\
= a^2 J_0 (ka) - \frac{2a}{k} \sum_{k=0}^\infty \frac{(-1)^k}{k! (k+m)!} \frac{1}{2^{2k+m}} x^{2k+m} \\
= a^2 J_0 (ka) - \frac{2a}{k} \sum_{k=0}^\infty \frac{(-1)^k}{k! (k+1)!} \frac{1}{2^{2k+1}} x^{2k+1} \\
= a^2 J_0 (ka) - \frac{2a}{k} J_1 (ka) 
\]
The magnetic moment \((\mu_2)\) becomes

\[
\mu = -\frac{n}{4} \frac{1}{J_0(ka)} \left[ a^2 J_0(ka) - \frac{2a}{k} J_1(ka) \right]
\]

\[= -\frac{n}{4} a^2 \left[ 1 - \frac{2}{ka} \frac{J_1(ka)}{J_0(ka)} \right]
\]

so that the magnetic polarizability is given by

\[
\chi = \frac{\mu}{\pi a^2 n} = -\frac{1}{4\pi} \left[ 1 - \frac{2}{ka} \frac{J_1(ka)}{J_0(ka)} \right]
\]

The longitudinal magnetic polarizability is half the transverse value calculated in problem 2.
The skin effect

distrib'tion of current density x-section of conductor in which non-zero variable current is flowing.

should expect that as frequency increases, the current will tend to be concentrated near the surface of the conductor: skin effect.

exact soln for current distib'n depends on shape of conductor & on the nature of the EM field which induces the current.

in a Thin wire the current distib'n is independent of the manner of excitation.

- E-field parallel to axis.
- H is in a plane perpendicular to the axis.

\[ \nabla \times H = \frac{B}{\mu} \]

wire of circular cross section by symmetry

\[ E = \text{constant} \quad \text{over the surface of the wire} \quad \hat{z} \text{axis} \]

\[ \frac{\partial E_x}{\partial x} = 0 \quad \frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x} \quad \frac{\partial E_x}{\partial z} = \frac{\partial E_x}{\partial x} \]

here \( E_x = 0 \) for all \( z \)

\( E_x \) does't change w/ change in \( x \) because \( \frac{\partial E_x}{\partial x} = 0 \)

non-zero \( E_y \) along \( x \) axis would violate cylindrical symmetry, so \( \frac{\partial E_y}{\partial x} = 0 \)

\[ \Rightarrow E_y = 0 \quad \text{all along} \quad x \text{ axis} \]

\[ \frac{\partial E_y}{\partial x} = 0 \quad \text{on} \quad x \text{ axis} \quad \Rightarrow \quad \frac{\partial E_x}{\partial y} = 0 \]

\[ \frac{\partial E_x}{\partial x} = 0 \quad , \quad \frac{\partial E_x}{\partial y} = 0 \quad , \quad \frac{\partial E_x}{\partial z} = 0 \]

\( \Rightarrow E_x = 0 = \text{const} \)
same argument applies to \( E_y \).

\[
\frac{\partial E_z}{\partial z} = 0, \quad \frac{\partial E_x}{\partial x} = \frac{\partial E_y}{\partial y} = 0, \quad \frac{\partial E_x}{\partial z} = \frac{\partial E_y}{\partial z} = 0
\]

\[\Rightarrow E_z = \text{const} \]

\[\therefore E = \text{constant throughout space external to the wire.} \]

\[
\begin{align*}
\text{outside} & \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = 0 \\
\text{inside} & \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{j}
\end{align*}
\]

The magnetic field outside must be the same as it would be outside a wire carrying a constant current equal to the instantaneous value of the variable current.

Inside the wire:

\[(58,1) \quad \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} \quad (\mu = \mu_0) \]

\[
\begin{align*}
\nabla \cdot \mathbf{H} &= 0 \\
\nabla \times \mathbf{H} &= \frac{4\pi}{c} \mathbf{j} \\
\mathbf{j} &= \sigma \mathbf{E} \\
&= \frac{4\pi \sigma}{c} \mathbf{E}
\end{align*}
\]

\[
\nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c} \frac{2}{\partial t} \nabla \times \mathbf{H} = -\frac{1}{c} \frac{4\pi}{c} \frac{2}{\partial t} \mathbf{E}
\]

\[
\nabla \cdot (\nabla \cdot \mathbf{E}) = \nabla^2 \mathbf{E} = \frac{4\pi \sigma}{c^2} \frac{\partial \mathbf{E}}{\partial t}
\]

\[
\nabla^2 \mathbf{E} = \frac{4\pi \sigma}{c^2} \frac{\partial \mathbf{E}}{\partial t}
\]

\[
\mathbf{E} = \frac{\partial \mathbf{E}}{\partial t}
\]

\[
\text{cylindrical polar coords}
\]

\[
\text{uncompensated change}
\]

\[
\text{no change}
\]

\[
\nabla \cdot \mathbf{E} = 0
\]
In cylindrical coordinates for a periodic field with frequency ω, \( E_2 \sim e^{-i\omega t} \)

\[
\left( \nabla^2 E_2 \right)_z = \nabla^2 E_2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial E_2}{\partial r} \right) = \frac{4\pi \sigma}{c^2} i\omega E_2
\]

since \( E_2 \) is independent of \( \phi \).

\[
= -\frac{4\pi \sigma \omega}{c^2} \frac{i\nu_2}{\nu_2} e^{-i\omega t} E_2
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial E}{\partial r} \right) + \kappa^2 E = 0
\]

\[
k = \frac{\sqrt{4\pi \sigma \omega}}{c} \frac{i\nu_4}{\nu_4}
\]

\[
= \frac{\sqrt{2\pi \sigma \omega}}{c} (1+i) = \frac{1+i}{\delta}
\]

**Penetration depth**

\[
\delta = \frac{c}{\sqrt{2\pi \sigma \omega}} \quad (as \ in \ eq. \ 59.4)
\]

**Dimensionless variable**

\[
x = kr
\]

\[
\frac{1}{x} \frac{d}{dx} x \frac{dE(x)}{dx} + E(x) = 0
\]

\[
\frac{d^2E}{dx^2} + \frac{1}{x} \frac{dE}{dx} + E = 0 \quad \text{Bessel DE w/ } m=0
\]

Solve: The solution that remains finite at origin is \( J_0(x) \)

(\text{Neumann function } N_0(x) \text{ diverges logarithmically at the origin})

\[
E = E_2 = \text{const } J_0(kr) e^{-i\omega t}
\]

The current density

\[
j = \sigma E \text{ is similarly distributed.}
\]
\[ \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} \]

(\mu = 1)

Here,

\[ \frac{\partial E_z}{\partial r} = -\frac{1}{c} \frac{\partial H_z}{\partial t} = \frac{1}{c} \frac{\partial}{\partial t} \mathbf{H}_z \]

\[ H_z = -\frac{c}{(\omega)} \frac{\partial E_z}{\partial r} = \frac{ic}{\omega} \frac{2}{\omega} \text{ const } j_0(kr)e^{-i\omega t} \]

\[ 2 J'_u = J_{u-1} - J_{u+1} \]

\[ 2 J'_0 = J_{-1} - J_1 = -2J_1 \]

\[ J_0 = -J_1 \]

\[ H_\phi = \frac{ic}{\omega} \text{ const } (-i) j_1(kr) e^{-i\omega t} \]

\[ = -i \text{ const } \frac{c}{\omega} j_1(kr) e^{-i\omega t} \]

\[ \frac{c}{\omega} \sqrt{\frac{4\pi \sigma}{c^2}} = \sqrt{\frac{4\pi \sigma}{\omega}} \]

\[ H_\phi = -i \text{ const } \sqrt{\frac{4\pi \sigma}{\omega}} j_1(kr) e^{-i\omega t} \]
The constant is determined as follows:

\[ D \times H = \frac{4\pi}{c} \mathbf{j} \]

Stokes' Theorem:

\[ \int \mathbf{da} \cdot D \times H = \oint \mathbf{dl} \cdot H = \frac{4\pi}{c} \int \mathbf{da} \cdot \mathbf{j} = \frac{4\pi I}{c} \]

\[ 2\pi a \mathbf{H}_0 = \frac{4\pi}{c} \mathbf{I} \]

\[ \mathbf{H}_0(r=a) = \frac{2}{ac} \mathbf{I} \]

\[ = \text{const} \sqrt{\frac{4\pi \sigma}{\omega}} J_1(ka) \]

\[ \Rightarrow \text{const} = \frac{\sqrt{\frac{\omega}{4\pi \sigma}} 2}{ac} \frac{I}{J_1(ka)} \]

Low frequency limit:

\[ a/\delta << 1 \quad \text{(large penetration compared with radius)} \]

\[ J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} \]

\[ J_1(x) = -\frac{\partial J_0}{\partial x} = -\left\{ -\frac{x}{2} + \frac{x^3}{16} - \frac{x^5}{384} \right\} \]

\[ E = \text{const} \left( 1 - \frac{\mathbf{r}^2}{4} + \frac{\mathbf{r}^4}{64} \right) e^{-i\omega t} \]

\[ k = \frac{1+i}{\delta} = \sqrt{2} e^{i\pi/4} \]

\[ k^2 = -\frac{2i}{\delta^2} \]

\[ k^4 = -\frac{4}{\delta^4} \]
High frequencies $a/\delta >> 1$, $kr >> 1$ over most of the x-section.

\[ J_0 \left( \frac{u \sqrt{2} i}{\delta} \right) \sim u^{-\frac{1}{2}} e^{(1-i)u} \]

\[ J_0 \left( \frac{kr}{\delta} \right) = J_0 \left( \frac{u \sqrt{2} i}{\delta} \right) \sim e^{(1-i)\frac{u}{\delta}} \quad \text{(keeping only rapidly varying exponential)} \]

\[ E_z = \text{const} \times e^{(1-i)\frac{t}{\delta}} e^{-iwt} \]

\[ E_z = \text{const}' e^{-(a-r)/\delta} i(a-r)/\delta e^{-iwt} \quad (60.7) \]

\[ H_{\phi} = -\text{const} \times i \sqrt{\frac{4\pi \sigma}{\omega}} J_1(kr) e^{-iwt} \]

\[ = -\frac{c}{i\omega \sigma} E_z = -\frac{c}{i\omega} \text{const}' \left( \frac{1}{\delta} - \frac{i}{\delta} \right) e^{-(a-r)/\delta} i(a-r)/\delta e^{-iwt} \]

\[ = \text{const}' \left( \frac{ic}{\omega \delta} + \frac{\omega}{\omega \delta} \right) e^{-(a-r)/\delta} i(a-r)/\delta e^{-iwt} \quad (60.7) \]

\[ H_{\phi} = \text{const}' (1+i) \sqrt{\frac{4\pi \sigma}{\omega}} e^{-(a-r)/\delta} i(a-r)/\delta e^{-iwt} \]

\[ \frac{c}{\omega \delta} = \frac{c}{\omega} \sqrt{\frac{4\pi \sigma}{\omega}} = \sqrt{\frac{4\pi \sigma}{\omega}} \]
The complex resistance

$\mathcal{E}(t) = R \mathcal{J}(t)$ at low frequency.

Resistance of wire at constant current.

Previous instant:

$\mathcal{J} = \hat{2}^{-1} \mathcal{E}$ or

$\mathcal{E}' = \hat{2} \mathcal{J}$

Given instant $t$.

Later instant $t^*$.

(61.2)

$\mathcal{E}'(t) = \mathcal{E}(w) e^{-iwt}$

$\mathcal{J}(t) = \mathcal{J}(w) e^{-iwt}$

$\mathcal{E}(w) e^{-iwt} = \int_0^\infty e^{-(\tau - t)} \mathcal{J}(\tau) e^{-i\omega \tau} d\tau = \int_0^\infty e^{-(\tau' + t)} \mathcal{J}(\tau') e^{-i\omega \tau'} d\tau'$

$\tau' = \tau - t$

$\tau = \tau' + t$

Since $\mathcal{E}(\tau' < 0) = 0$

$\mathcal{E}(w) = \mathcal{J}(w) \int_{-\infty}^{\infty} e^{-i\omega \tau'} d\tau'$

referred to as $\mathcal{Z}(w)$ [rather than $\mathcal{Z}(-w)$?]

Complex resistance or impedance of the conductor.
\[ \mathcal{E} = \int_{t}^{t+T} f(t) \, dt \]

emf at a given instant

all subsequent instants

\[ \mathcal{E}(t) = \int_{t}^{\infty} g(t') J(t') \, dt' \]

\[ \tau' = t - \tau \quad d\tau = d\tau' \]

\[ \tau = \tau' + t \]

\[ \mathcal{E}(w) e^{-j\omega t} = \int_{-\infty}^{\infty} g(t') J(w) e^{-j\omega t} \]

\[ \mathcal{E}(w) = J(w) \int_{-\infty}^{\infty} g(t') e^{-j\omega t'} \]

defined as \( Z(w) \), though the conventional definition of the F.T. would prefer \( Z(-w) \).

\[ \mathcal{E}(0) = J(0) Z(0) \]

must correspond to \( R \) in eq. 61.1.

\[ \Rightarrow R \text{ is the zero-order term in an expansion of } Z(w) \text{ in powers of } w. \]

To find the next term, we must take into account both \( R \) and the self-inductance \( L \) of the wire.

Consider a linear circuit containing a variable emf \( \mathcal{E}(t) \).

work done per unit time by the electric field on charges moving in the wire is

\[ \mathcal{E}J \text{. This work goes partly into Joule heat and partly to change } \Phi \text{ energy of the magnetic field of the current. Joule heat per unit time is } R J^2 \text{ magnetic energy is } \frac{LJ^2}{2c^2}. \]
\[ E = \frac{R J^2 - \frac{d}{dt} \frac{L J^2}{2c^2}}{R J^2 + \frac{L J}{c^2} \frac{dJ}{dt}} = \frac{E^0 - i\omega L}{c^2} J_0 e^{-i\omega t} \]

**Complex Monochromatic Forms**

\[ E = E_0 e^{-i\omega t} \]

\[ J = J_0 e^{-i\omega t} \]

\[ E_0 = \frac{R J_0 - i\omega L}{c^2} J_0 \]

\[ E = \left\{ \begin{array}{l} R - \frac{i\omega L}{c^2} \frac{d}{dt} \frac{L J}{c^2} \quad (\text{having multiplied by} \quad e^{-i\omega t}) \\
\equiv \frac{R}{Z(\omega)} \end{array} \right. \]

\[ J_0 e^{-i\omega t} = \frac{E_0 e^{-i\omega t}}{R - \frac{i\omega L}{c^2}} = \frac{E_0 e^{-i\omega t + i\phi}}{\sqrt{R^2 + \frac{L^2}{c^4}}}, \quad \text{where} \quad \phi = \tan^{-1} \frac{\omega L}{c^2 R} \]

\[ J(t) = \text{Re} \frac{E_0 e^{-i(\omega t + i\phi)}}{\sqrt{R^2 + \frac{L^2}{c^4}}} = \frac{E_0}{\sqrt{R^2 + \frac{L^2}{c^4}}} \cos(\omega t - \phi) \]

\[ \text{b1.6} \]

\[ \uparrow \text{assuming that the time origin is chosen so that} \quad E_0 \text{ is real} \]
\[ Z = R - \frac{1}{2} \mu L \] (61.5)

\( J \)

resistance determines energy dissipation in the circuit.

This relation is general.

\[ EJ = \text{power required to maintain a periodic current} \]

\[ E = \frac{1}{2} (\varepsilon_0 e^{-iwt} + \varepsilon_0^* e^{iwt}) \]

\[ J = \frac{1}{2} (\varepsilon_0 e^{-iwt} + \varepsilon_0^* e^{iwt}) \]

\[ \overline{EJ} = \frac{1}{4} (\varepsilon_0 \varepsilon_0^* + \varepsilon_0^* \varepsilon_0) = \frac{1}{2} \text{Re} \varepsilon_0 \varepsilon_0^* \]

\[ \text{cycle-averaged energy dissipation} \]

expressed in complex form:

\[ \overline{E} = J^* \]

\[ \frac{1}{2} \text{Re} (Z J J^*) = \frac{1}{2} Z' |J|^2 \]

Real function:

\[ J(t) = \frac{1}{2} (J + J^*) \]

\[ \overline{J(t)} = \frac{1}{4} (J - J^* + J^* - J) \]

\[ = \frac{1}{2} |J|^2 \]

\[ Q = Z' \overline{J^2(t)} \] (61.8)

since \( Q \) is positive, \( Z'(\omega) \) is positive.

\( Z(\omega) \) for a wire of circular cross-section for any frequency which satisfies the quasi-steady condition, i.e. without neglecting the skin effect.

\[ EJ = \text{change in magnetic field energy outside + total energy consumed inside the wire (changing field + internal heat)} \]

\[ \overline{EJ} = \frac{d}{dt} \left( \frac{L e J^2}{2 \varepsilon} \right) + \frac{CEH}{4\pi^2} \]

\[ \overline{EJ} = \frac{L e J^2}{2 \varepsilon} \frac{dJ}{dt} + \frac{CEH}{2} \overline{J^2(t)} \]
\[
EJ = \frac{LeJ \partial \phi}{c^2 \partial t} + \frac{E \partial J}{\partial t}
\]

\[
E = \|E\| + \frac{Le}{c^2} \frac{\partial J}{\partial t}
\]

Linear approx can use complex quantities

\[
E = 2J = \|E\| - \frac{i\|E\|}{c^2} \frac{\partial J}{\partial t}
\]

\[
Z = \frac{2\|E\|}{ca} \frac{\partial J}{\partial t} - \frac{i\|E\|}{c^2} \frac{\partial J}{\partial t}
\]

For general frequencies

(5.2) \quad E = \text{const} \cdot J_0(kr) e^{-i\omega t}

(5.4) \quad H = -\text{const} \cdot \sqrt{\frac{\mu_0}{\omega}} J_1(kr) e^{-i\omega t}

\[
Z = \frac{2\|E\|}{ca} \frac{\partial J}{\partial t} - \frac{i\|E\|}{c^2} \frac{\partial J}{\partial t}
\]

\[k = \frac{1+i}{c} \sqrt{2\pi \omega}
\]

\[
\frac{1+i}{c} = \frac{k}{\sqrt{2\pi \omega}}
\]

\[
Z = \frac{\|E\|}{2\pi \alpha} \frac{k}{\sqrt{2\pi \omega}} J_0(ka) - \frac{i\|E\|}{c^2} \frac{\partial J}{\partial t}
\]

\[
R = \frac{\|E\|}{\pi \alpha \omega}
\]
In the low frequency limit (weak skin effect, large penetration depth) we can keep just a few low-order terms in the power series expansions of the Bessel functions.

\[ J_0(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!k!} \frac{x^{2k}}{2^{2k}} \]

\[ J_0(x) \approx 1 - \frac{x^2}{4} + \frac{x^4}{64} \] (OK)

\[ J_1(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!(k+1)!} \frac{x^{2k+1}}{2^{2k+1}} \]

\[ = \frac{x^2}{2} - \frac{x^3}{16} + \frac{x^5}{384} \]

\[ J_1(x) = -\frac{x}{2} \left( 1 - \frac{x^2}{8} + \frac{x^4}{192} \right) \] (OK)

\[ -\frac{1}{32} + \frac{1}{64} + \frac{1}{192} = -\frac{1}{192} \]

\[ 1 = (1+\alpha) \leq 1 \Rightarrow \frac{1}{1+\alpha} = 1 - \alpha + \alpha^2 \]

\[ \frac{J_0(x)}{J_1(x)} = \left( 1 - \frac{x^2}{4} + \frac{x^4}{64} \right) \left( \frac{2}{x} \right) \left( 1 + \frac{x^2}{8} - \frac{x^4}{192} + \frac{x^4}{64} \right) \]

\[ = \frac{2}{x} \left( 1 - \frac{x^2}{8} - \frac{x^4}{192} \right) \] (lookin' good)
Back to the impedance,

\[ Z = -\frac{i\omega}{c^2} L e + \frac{1}{2} R \hat{k} a \frac{Z}{k a} \left( 1 - \frac{k^2 a^2}{8} - \frac{k^4 a^4}{192} \right) \]

\[ = -\frac{i\omega}{c^2} L e + R \left( 1 - \frac{k^2 a^2}{8} - \frac{k^4 a^4}{192} \right) \]

\[ k = \frac{1 + i}{\delta} = \frac{\sqrt{2}}{\delta} e^{i\pi/4} \]

\[ Z = -\frac{i\omega}{c^2} L e + R \left( 1 - \frac{i}{4} \left( \frac{a}{\delta} \right)^2 + \frac{1}{48} \left( \frac{a}{\delta} \right)^4 \right) \]

In particular,

\[ Z'(\omega) = \text{Re}(Z) = R \left( 1 + \frac{1}{48} \left( \frac{a}{\delta} \right)^4 \right) \]

\[ \delta = \frac{c}{\sqrt{2\pi\sigma\omega}} \quad \left( \frac{a}{\delta} \right)^4 = \frac{a^4 (2\pi\sigma\omega)^2}{c^4} \]

\[ Z'(\omega) = R \left( 1 + \frac{a^4 (2\pi\sigma\omega)^2}{12 c^4} \right) \]

This expression incorporates a second-order-in-\(\omega\) increase in the rate of dissipative loss due to the skin effect—this is missing from the lower-order expression (61.5).
In the case of strong skin effect (high frequency) we can use

\( Z = - \frac{i \omega}{c^2} L_e + \frac{\sigma}{\epsilon a} \frac{1}{1 + i} \sqrt{\frac{\omega}{2 \pi \sigma}} \)

\[ Z' = \frac{\sigma}{\epsilon a} \sqrt{\frac{\omega}{2 \pi \sigma}} \quad Z'' = - \frac{i \omega}{c^2} L_e - \frac{\sigma}{\epsilon a} \sqrt{\frac{\omega}{2 \pi \sigma}} \]

\[ Z'' = - \frac{i \omega}{c^2} \left[ L_e + \frac{\sigma L \delta}{\epsilon a} \sqrt{\frac{1}{2 \pi \sigma \omega}} \right] \]

\[ Z'' = - \frac{i \omega}{c^2} \left[ L_e + \frac{\sigma L \delta}{\epsilon a} \right] \]

\[ L_i = \frac{\epsilon}{2} \]

Dimension check:

\( \delta = \frac{c}{\sqrt{2 \pi \sigma \omega}} \)

(59.4) have dimensions of length?

Does induction \( L \) have units of length?

Conductivity \( \sigma = \frac{|j|}{E} = \frac{\text{charge per unit time per unit area}}{\text{electric field}} \)

\( \sigma \) = \( \frac{c \text{ m s}^{-1}}{\text{m} \text{ s}^{-1} \text{ m}^{-1}} = \frac{1}{\text{time}} \) as expected.
\[ \mathcal{L}_{aa} = \frac{L_{aa}}{c^2} \]

\[ \text{self inductance} \]

\[ [L] = \frac{\text{energy}}{\text{charge}^2 \text{s}^{-2}} = \frac{\text{cm}^2}{\text{cm}} = \text{cm} \]

\[ \left[ \frac{e^2}{r} \right] = \text{energy} \]

returning to weak-skin-effect case (low frequency) (41.1a)

\[ Z' = R \text{ if } \left( \frac{\pi \sigma w a^2}{c^2} \right)^2 \ll 12 \]

we also have

\[ \left| \frac{Z''}{Z'} \right| = \frac{\omega \ell e}{c^2 R} \approx \frac{\omega}{c^2 R} \]

\[ \approx \frac{\omega}{c^2 R} \frac{2 \pi e \ln (2/a)}{2} \]

\[ (34.1) \]

\[ \frac{\pi \sigma w a^2}{c^2} \ll 12 \Rightarrow \frac{\pi \sigma w a^2}{c^2} \ll 2 \sqrt{3} \]

\[ R = \frac{2}{\pi \sigma w} \quad (\text{below 61.11}) \]

\[ \text{Use of (61.5) to take account of the effect of self-inductance on the emf requires that} \]

\[ \frac{\omega}{c^2 R} \ll \frac{\pi \sigma w a^2}{c^2} \ll 2 \ln (2/a) \]

\[ \text{with } \ll 2 \sqrt{3} \]

\[ \text{unnecessary slash in book's expression} \]

\[ \text{if } \text{low frequency is high} \]
In practice, however, the most important case is that in which the self-inductance is large compared with that of an uncoiled wire (hence \( L_s \gg L_r \), straight, \( L_i \)). Here the skin effect, associated with \( L_i \), cannot matter. In such circuits, (61.5) (61.4 with constant \( L \frac{1}{R} \)) can be used over a fairly wide range of frequencies.

Circuit in a variable external magnetic field \( H_e \):

\( E_e \) is the variable which would be induced by \( H_e \) in the absence of conductors:

\[
D \times E = -\frac{1}{c} \frac{\partial B}{\partial t} = -\frac{1}{c} \frac{\partial H}{\partial t}
\]

\( \nabla = \) everywhere.

\( H_e \) and \( E_e \) vary only slightly over the thickness of a wire (unlike the fields of the current in the wire).

Circulation of \( E_e \) is the emf induced in the circuit by the variable magnetic field:

\[
E = \oint E_e \cdot dl = \oint d\tau \cdot D \times E_e = -\frac{1}{c} \oint d\tau \cdot \frac{\partial B_e}{\partial t} = -\frac{1}{c} \oint d\tau \cdot \frac{\partial H}{\partial t}
\]

From 61.4

\[
E = RJ + \frac{1}{c^2} L \frac{dJ}{dt}
\]

\[
RJ + \frac{L}{c^2} \frac{dJ}{dt} = -\frac{1}{c} \frac{d\Phi_e}{dt}
\]

\[
RJ = -\frac{1}{c} \left( \frac{d\Phi_e}{dt} + \frac{L}{c} \frac{dS}{dt} \right) = \frac{1}{c} \frac{d\Phi_e}{dt}
\]
If the self-inductance is a function of time due to the shape of the circuit changing, (6.1.14) becomes

\[ RL = -\frac{1}{2}\frac{d}{dt}\frac{\Phi_e}{e} - \frac{1}{2}\frac{d}{dt}\frac{1}{e^2}\frac{d^2}{dt^2}LJ \]

(6.1.15)

*Total error: Part due to changing external magnetic field!
Changing field induced by the current in the circuit.*

**Why is \( \frac{LJ}{c} \) the flux due to current in the circuit?**

\[(33.2) \quad \Phi_e = \int \frac{A \cdot dV}{2c} = \frac{LJ^2}{2c^2} \quad \text{single conductor, no external field} \]

\[ \text{all three are parallel} \]

\[ \frac{1}{2e} \int \frac{A \cdot dA}{d^2} \]

\[ = \frac{1}{2e} \oint A \cdot dl \int da \cdot \phi \]

\[ \int \text{everywhere along the wire} \]

\[ = \frac{LJ^2}{2c^2} \Phi_{\text{induced}} \]

\[ H_{\text{induced}} \quad \text{(since there are no external sources)} \]

\[ \frac{LJ^2}{2c^2} = \frac{LJ}{2c} \Phi_{\text{induced}} \Rightarrow \frac{LJ}{c} = \Phi_{\text{induced}} \]

If there are several circuits in proximity, carrying currents \( J_a \), then for each of them \( \Phi_e \) in (6.1.14) is the sum of the magnetic fluxes due to all other circuits (and no external field, if any).

The magnetic flux through \( \Phi_{ab} \) the circuit due to the current \( J_b \) is \( \text{lab } J_b / c^2 \).

**Here's why:**

\[(33.5) \quad \Phi_{ab} = \int J_a \cdot A_b dV / e = \text{lab } J_a J_b / c^2 \]

\[ \Phi = \frac{1}{e} \oint A_b \cdot dA_a \int J_a dV / e_a \]

\[ \frac{dA_a}{da} \quad \frac{dV}{da} \quad \text{both} \parallel \text{to} \quad \frac{dV}{da} \]

\[ = \frac{J_a}{c} \oint dV_a \cdot \nabla \times A_b = \frac{J_a}{c} \Phi_{ab} \]

\[ \text{by} \quad \nabla \times H_b \Rightarrow \Phi_{ab} = \frac{\text{lab } J_b}{c} \]
For the variable currents in the circuit:

\[ p_a J_a + \frac{1}{c^2} \sum \limits_{b} L_{ab} \frac{dS_b}{dt} = \varepsilon_a \]

The emf produced in the \( a \)th circuit by sources external to it.
6.2 Capacitance in a quasi-steady current circuit

- Variable current can flow in an open circuit.
- Linear circuit whose ends are connected to the plates of a capacitor.
- Variable current periodically changes and discharges the capacitor.

- Distance between the plates is small.
- Magnetic energy $\frac{L J^2}{2C^2}$ is self-inductance of closed circuit obtained by joining the capacitor plates by a short piece of wire (neglect skin effect).

\[(6.1.4) \quad E = RJ + \frac{1}{C^2} L \frac{dJ}{dt} + \frac{e}{C} \]

- The voltage drops across the resistance and the potential difference across the capacitor plates.

\[
\frac{R}{\text{MN}} \quad \frac{\text{c/d}}{\text{LJ}} \quad \frac{\text{v}^2}{\text{1}}
\]

\[E = RJ + \frac{1}{C^2} L \frac{dJ}{dt} + \frac{e}{C} \quad J = \frac{de}{dt} \]

\[E = R \frac{de}{dt} + \frac{1}{C^2} L \frac{d^2e}{dt^2} + \frac{e}{C} \quad (6.2.1)\]

- The voltage is complex in general.
- The current is complex in general.

\[\mathcal{E} = \mathcal{E}_0 e^{-i\omega t} \quad \mathcal{E} = \mathcal{E}_0 e^{-i\omega t} = -i\omega \mathcal{R} e - \frac{w^2 L e}{C} + \frac{e}{C} \quad J = \frac{de}{dt} = -iwe \]

\[\mathcal{E} = \mathcal{R} J - \frac{w^2 L J}{C} \quad \mathcal{E} = (R - i\omega L + \frac{i}{Cw}) J \]

\[Z = R - i \left( \frac{wL}{C} - \frac{1}{wC} \right) \quad (6.2.2)
\]

\[\text{Dec 06} \]
\[ J(t) = \text{Re}\left(\frac{e^{i\omega t}}{R - iK}\right) = \text{Re}\left(\frac{\epsilon_0 e^{-i\omega t}}{R\epsilon^2 - i\omega C}\right) \]

where \( K = \frac{\omega L}{c^2} - \frac{i}{\omega C} \)

\[ R + iK = \sqrt{R^2 + \epsilon^2} e^{i\phi} \]

\[ \phi = \tan^{-1}(K/R) \]

\[ \frac{1}{R^2 + K^2} \text{Re}\left(\epsilon_0 e^{-i\omega t + \phi}\right) = \frac{\epsilon_0}{\sqrt{R^2 + \epsilon^2}} \cos(\omega t - \phi) \]

Given the current in a circuit to which an external emf \( E = \epsilon_0 \cos(\omega t) \) is applied.

If \( \epsilon = 0 \), the current in the circuit consists of free electric oscillations.

\[ \epsilon = J \Rightarrow \frac{1}{Z} = 0 \Rightarrow \frac{1}{R} = 0; \]

\[ 0 = R\omega - \frac{i}{\epsilon^2} - \frac{1}{C} \]

\[ 0 = \omega^2 \frac{L}{c^2} + iR - \frac{1}{C} \]

\[ \omega = -iR \pm \sqrt{-R^2 + \frac{4L}{c^2C}} \]

\[ \frac{2L}{c^2} \]

\[ w = -i \frac{Rc^2}{2L} \pm \frac{c^2}{2L} \sqrt{\frac{4L}{c^2C} - R^2} \]

\[ w = -i \frac{Re^2}{2L} \pm \sqrt{\frac{\epsilon^2}{LC} - \left(\frac{c^2}{2L}\right)^2} \]

\[ (6.2.4) \quad 0 \text{ Dec 06} \]
If \( \frac{e^2}{KC} > \left( \frac{Re^2}{2L} \right)^2 \),

\[
4L > CR_e^2
\]

periodic oscillations damped by decrement

\[
\Rightarrow \frac{Re^2}{2L}
\]

DAMPING RATE INDEPENDENT OF C

\( R \to 0 \) undamped oscillations

\[
w = \frac{e}{\sqrt{LC}} \quad \text{Thomson's formula (1853!)}
\]

generalizing (62.1) to a system of several inductively coupled circuits containing

capacitors \( \mathbf{J}_e = \mathbf{d} \mathbf{e}/\mathbf{d} \mathbf{t} \)

\[
\mathbf{E}_e = \frac{1}{c^2} \sum_{\mathbf{b}} \mathbf{L}_{\mathbf{ab}} \frac{\mathbf{d}^2 \mathbf{e}_b}{\mathbf{d}t^2} + \frac{\mathbf{R}_e \mathbf{d} \mathbf{e}_a}{\mathbf{d} \mathbf{t}} + \frac{\mathbf{e}_a}{\mathbf{C}_a}
\]

monochromatic currents \( \mathbf{J}_e = -i \omega \mathbf{e}_a \quad \mathbf{e}_a = \frac{i}{\omega} \mathbf{J}_a \)

\[
\mathbf{E}_a = \frac{1}{c^2} \sum_{\mathbf{b}} \mathbf{L}_{\mathbf{ab}} (-\omega^2) \frac{i}{\omega} \mathbf{J}_b + \frac{\mathbf{R}_e \mathbf{J}_a}{\mathbf{C}_a} + \frac{i \mathbf{J}_a}{\mathbf{C}_a}
\]

\[
= \sum_{\mathbf{b}} \mathbf{Z}_{\mathbf{ab}} \mathbf{J}_b
\]

\[
\mathbf{Z}_{\mathbf{ab}} = -i \frac{\mathbf{L}_{\mathbf{ab}} \omega}{\mathbf{c}^2} + \delta_{\mathbf{ab}} \left( \frac{\mathbf{R}_e}{\mathbf{C}_a} + \frac{i}{\omega \mathbf{C}_a} \right)
\] (62.8)

Eigenfrequencies of the current system are given

\[
\text{det} [ \mathbf{Z}_{\mathbf{ab}} ] = 0
\]

\[
\sum_{\mathbf{b}} \frac{\mathbf{L}_{\mathbf{ab}}}{\mathbf{c}^2} \mathbf{e}_b + \mathbf{R}_e \mathbf{e}_a + \frac{\mathbf{e}_a}{\mathbf{C}_a} = \mathbf{E}_a
\]

Lagrangian

\[
\mathbf{L} = \sum_{\mathbf{a}, \mathbf{b}} \frac{1}{2 \mathbf{c}^2} \mathbf{L}_{\mathbf{ab}} \mathbf{e}_a \mathbf{e}_b - \sum_{\mathbf{a}} \frac{\mathbf{e}_a^2}{2 \mathbf{C}_a} + \sum_{\mathbf{a}} \mathbf{e}_a \mathbf{E}_a
\]

dissipative function

\[
\frac{d}{dt} \frac{\partial \mathbf{L}}{\partial \mathbf{\dot{e}_a}} = \frac{1}{\mathbf{c}^2} \sum_{\mathbf{b}} \mathbf{L}_{\mathbf{ab}} \mathbf{\dot{e}_b} - \frac{\partial}{\partial \mathbf{\dot{e}_a}} \frac{\mathbf{F}_\mathbf{a}}{\mathbf{E}_a} + \frac{\mathbf{e}_a}{\mathbf{C}_a} - \mathbf{\dot{e}_a}
\]

1st term \( \frac{\partial}{\partial \mathbf{\dot{e}_a}} \mathbf{R}_e \mathbf{\dot{e}_a} \)

2nd term \( \frac{\partial}{\partial \mathbf{\dot{e}_a}} \mathbf{R}_e \mathbf{\dot{e}_a} \)
Discuss the properties of electric oscillators in a circuit consisting of an infinite succession of identical meshes containing impedances.

\[ Z_j = -i \left( \frac{1}{\omega C_j} \right) \quad j = 1, 2 \]

There are no external emf's, and so, adding products of current x impedance around the \( n \)th mesh, we get

\[-Z_2 i_{\alpha-1} + Z_2 i_{\alpha} + Z_1 i_{\alpha} + Z_2 i_{\alpha} - Z_2 i_{\alpha+1} = 0\]

or

\[ Z_1 i_{\alpha} + Z_2 (2 i_{\alpha} - i_{\alpha-1} - i_{\alpha+1}) = 0 \]

This is a linear difference equation in the integral variable \( \alpha \), with constant coefficients.

Seek a solution \( i_{\alpha} = \text{const} \times q^\alpha \)

\[ Z_1 q^\alpha + Z_2 (2 q^\alpha - q^{\alpha-1} - q^{\alpha+1}) = 0 \]

or

\[ q^2 - q \left( \frac{Z_2}{Z_1} + 2 \right) + 1 = 0 \]
The solutions are

\[ q = 1 + \frac{z_1}{z_2} \pm \frac{1}{2} \sqrt{(1 + \frac{z_1}{z_2})^2 - 4} \]

\[ q = 1 + \frac{z_1}{z_2} \pm \sqrt{(1 + \frac{z_1}{z_2})^2 - 1} \]

The solutions are **real** if

\[(1 + \frac{z_1}{z_2})^2 \geq 4 \Rightarrow \left| \frac{z_1}{z_2} + 1 \right| \geq 1\]

i.e., \( \frac{z_1}{z_2} \geq 0 \) or \( \frac{z_1}{z_2} \leq -2 \)

\[ \therefore \text{Both solutions are real if} \]

\[ \frac{z_1}{z_2} \leq -4 \quad \text{or} \quad 0 \leq \frac{z_1}{z_2} \]

\[ \frac{Z_1}{Z_2} = \left( \frac{\frac{w}{c^2} L_1 - \frac{1}{w C_1}}{\frac{w}{c^2} L_2 - \frac{1}{w C_2}} \right) = \frac{w^2 L_1 - \frac{1}{C_1}}{w^2 L_2 - \frac{1}{C_2}} \]

\[ \frac{Z_1}{Z_2} \geq 0 \Rightarrow \frac{c^2}{L_1 C_1} \text{ and } \frac{c^2}{L_2 C_2} \]

**OR**

\[ \frac{c^2}{L_1 C_1} \text{ and } \frac{c^2}{L_2 C_2} \]

\[ \text{OR} \]

\[ \frac{w^2}{c^2} \leq \frac{c^2}{L_1 C_1} \text{ and } \frac{c^2}{L_2 C_2} \]
\[ \frac{Z_1}{Z_2} \leq -4 \quad \Rightarrow \quad w^2 \left( \frac{L_1}{c^2} - \frac{1}{C_1} \right) \leq -4 \left( w^2 \frac{L_2}{c^2} - \frac{1}{C_2} \right) \]

\[ w^2 \left( L_1 + 4L_2 \right) \leq \left( \frac{1}{C_1} + \frac{4}{C_2} \right) c^2 \]

(Provided \( w^2 > \frac{c^2}{L_2 C_2} \))

OR

\[ w^2 \left( L_1 + 4L_2 \right) \geq \left( \frac{1}{C_1} + \frac{4}{C_2} \right) c^2 \]

(Provided \( w^2 < \frac{c^2}{L_2 C_2} \)).

In any of these cases,

\[ q_+ q_- = \left( 1 + \frac{Z_1}{2Z_2} \right)^2 - \left( 1 + \frac{Z_1}{2Z_2} \right)^2 + 1 = 1. \]

One of these roots, \( q_1 \) say, is less than 1 in absolute magnitude, and \( q_2 \) is greater. The propagation of undamped oscillations in the circuit is then impossible.

Let \( L \) consider a large but finite circuit to illustrate this point. An initial oscillatory impulse is given at one end. The closure of the other end corresponds to a boundary condition that determines the ratio \( C_1 \) and \( C_2 \) in the general solution:

\[ i' = c_1 q_1 - (A-x) - c_2 q_2 - (A-x) \]

where \( A \) is the index of the end of the circuit.

\[ i' = c_1 \left( \frac{1}{q_1} \right)^{A-x} + c_2 q_2^{A-x} \]

\[ \text{where} \]

\[ i' = c_1 \left( \frac{1}{q_1} \right)^{A-x} + c_2 q_2^{A-x} \]
Both $\frac{1}{q_1}$ and $q_2$ have absolute magnitudes greater than one. Only the first term is non-negligible near the end of the circuit. The second term, which can be nonzero only near the beginning of the circuit, must damp out a short distance in.
**L 3. ECM § 63 Motion of a conductor in a magnetic field**

\[ \mathbf{j} = \sigma \mathbf{E} \] valid for conductors at rest.

Moving conductor (or part of it) stationary in frame \( \mathcal{K'} \). \( \mathbf{j} = \sigma \mathbf{E} + \frac{v}{c} \times \mathbf{B}/c \)

- Electric field in \( \mathcal{K'} \).
- Velocity of \( \mathcal{K'} \) relative to \( \mathcal{K} \).

2nd term is in general not small compared with the first term despite the factor \( \frac{v}{c} \).

For example, the E-field due to electromagnetic induction from a variable magnetic field contains a factor \( \frac{1}{c^2} \) compared with the magnetic field.

Rate of Joule heat per unit volume

\[ \frac{\mathbf{j}^2}{\sigma} = \mathbf{j} \cdot \sigma \mathbf{E}' = \mathbf{j} \cdot \left( \mathbf{E} + \frac{1}{c} \times \mathbf{v} \times \mathbf{B} \right) \]

"Effective" E-field producing the conduction current.

Emf acting on a closed linear circuit is

\[ E' = \oint (\mathbf{E} + \frac{v}{c} \times \mathbf{B}/c) \cdot d\mathbf{l} \]
\[
\mathcal{E} = -\frac{1}{c} \left( \frac{\partial \Phi_B}{\partial t} \right)_{\nu = 0} - \frac{1}{c} \oint \mathbf{dl} \cdot \mathbf{v} x \mathbf{B} \frac{\partial \mathbf{v}}{\partial t} \]

**Faraday's Law**

\[
\mathcal{E} = -\frac{1}{c} \frac{d\Phi_B}{dt}
\]
In a static magnetic field, the change in flux may reduce entirely to the motion of the circuit. If the circuit moves in such a way that every point in it moves along a line of force, then the flux through the circuit does not vary.

\[ \text{Flux thru closed surface is zero} \]
\[ \text{Flux thru side surface is zero} \]
\[ \text{Flux thru new surface is same as flux thru old} \]

To induce an emf, conductor must move so as to cross lines of magnetic force.

EM field in a moving conductor obeys

\[ \nabla \times \mathbf{E} = -\frac{1}{c^2} \frac{\partial \mathbf{B}}{\partial t} \]

\[ \nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{j} = \frac{4\pi}{c} \left( \mathbf{E} + \frac{1}{c} \nabla \times \mathbf{B} \right) \]

\[ \nabla \cdot \mathbf{B} = 0 \]

\[ \frac{c}{4\pi} \nabla \times \left( \frac{\nabla \times \mathbf{H}}{c} \right) - \frac{1}{c} \nabla \times (\nabla \times \mathbf{B}) = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \]

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\nabla \times \mathbf{B}) = \frac{c^2}{4\pi} \nabla \times \left( \frac{\nabla \times \mathbf{H}}{c} \right) \]

In a homogeneous cond. with const \( \sigma \) and mag permeability \( \mu \)

\[ \frac{\partial \mathbf{H}}{\partial t} = \nabla \times (\nabla \times \mathbf{H}) - \frac{c^2}{4\pi\mu} \nabla \times (\nabla \times \mathbf{H}) \]

\[ \frac{\partial \mathbf{H}}{\partial t} - \nabla \times (\nabla \times \mathbf{H}) = \frac{c^2}{4\pi\mu} \nabla^2 \mathbf{H} \]

\[ \nabla (\nabla \cdot \mathbf{H}) - \nabla \times \nabla \times \mathbf{H} = 0 \]
only one conductor moving as a whole in a magnetic field (without changing shape), use sys coords fixed in conductor, where cond is at rest and field varies in a given manner, new sys coords is not inertial in general.

Electromagnetic induction is independent of source of change in magnetic flux.

\[ \text{curl}(\vec{v} \times \vec{B}) = \nabla \times (\vec{v} \times \vec{B}) = \nabla \cdot \vec{B} \nabla \psi - \nabla \cdot \nabla \psi \vec{B} \]

\[ = (\nabla \cdot \nabla) \psi + (\nabla \cdot \vec{v}) \vec{B} - (\nabla \cdot \vec{B}) \vec{v} - \vec{v} \cdot \nabla \vec{B} \]

LHS of (63.6) becomes

\[ \frac{\partial \vec{B}}{\partial t} + \nabla \cdot \vec{v} \vec{B} - \vec{B} \cdot \nabla \vec{v} = \text{time derivative of } \vec{B} \text{ wrt axes fixed in the rotating body.} \]

"substantial" time derivative

\[ \frac{\partial \vec{B}}{\partial t} + \text{deriv at point moving w/ velocity } \vec{v} \text{ relative to the body.} \]

For a rotation \( \vec{v} = \vec{v} \times \), so

\[ \vec{B} \cdot \nabla \vec{v} = B_j \delta_{ij} \varepsilon_{kli} \frac{\partial \vec{B}_k}{\partial x_i} = B_j \varepsilon_{kli} \frac{\partial \vec{B}_k}{\partial x_i} \]

\[ = B_j \varepsilon_{kli} \frac{\partial \vec{B}_k}{\partial x_i} = (\nabla \times \vec{B})_i \]
unipolar induction occurs when a magnetized conductor rotates
stationary wire, two sliding contacts A and B

\[ E = \frac{1}{c} \oint (v \times B) \cdot dl = \frac{1}{c} \oint \left( (\omega \times l) \times B \right) \cdot dl 
\]

\[ = -\frac{1}{c} \oint (\omega \times l \times B) \cdot dl \]

because \( v = 0 \) for B0A

\( v = -\omega \times l \)

\[ E' = -\frac{1}{c} \oint \left( B \times (l \times \omega) \right) \cdot dl \]

\[ = -\frac{4}{c} \oint B \times (l \times \omega) \cdot dl \]

sign difference with (63.9)
emf due to unipolar induction from pole through a uniformly magnetized sphere rotating uniformly about the direction of magnetization.

Field is static and \( \mathbf{H} = \frac{4\pi J}{c} = 0 \) within the sphere.

From (63.6)
\[
\text{curl} \left( \nabla \times \mathbf{B} \right) = \frac{2B}{c} + \frac{c^2}{4\pi} \text{curl} \left( \frac{\mathbf{H}}{c^2} \right) = 0
\]

\[
\text{curl} \left( \nabla \times \mathbf{B} \right) = 0 \quad \text{inside sphere} \quad \nabla \times \mathbf{B} = 0 \quad \text{outside}
\]

\[
\mathbf{E} = \frac{1}{c} \int_{ACB} \mathbf{dl} \times (\nabla \times \mathbf{B}) = \frac{1}{c} \int_{AOB} \mathbf{dl} \times (\nabla \times \mathbf{B}) - \frac{1}{c} \int_{OB} \mathbf{dl} \times (\nabla \times \mathbf{B})
\]

Using \( \mathbf{dl} \)'s choice of sign?

\[
\mathbf{E} = \frac{1}{c} \int_{0}^{a} \mathbf{dl} \times (\nabla \times \mathbf{B}) = \frac{a^2}{2c} \mathbf{B}_0 \]

\[
\mathbf{E} = \frac{1}{c} \int_{0}^{a} \mathbf{dl} \times (\nabla \times \mathbf{B}) = \frac{a^2}{2c} \mathbf{B}_0
\]

\( \mathbf{E} \) and \( \mathbf{B} \) are parallel

\[
\text{magnetic induction inside sphere}
\]

\[
\mathbf{E} = \mathbf{dl} \times (\hat{r}_i \hat{B}_j \epsilon_{ijk} \hat{r}_k \hat{B}_m) = \mathbf{dl} \times (\hat{r}_i \hat{B}_j \hat{r}_j \hat{B}_i)
\]

\[
\mathbf{dl} : \hat{r}_i \hat{B}_j \hat{r}_j \hat{B}_i
\]
In a uniformly magnetized sphere (in the absence of an external field)

\[ B = H + 4\pi M \quad \text{in general} \]

In a uniformly magnetized sphere (in the absence of an external field), Nielson says

\[ B + 2H = 0 \quad \text{(with reference to Eq. (8.1))} \]

Section 8 deals w/ a dielectric ellipsoid in a uniform external electric field.

**Dielectric sphere**

\[ \Phi' - \text{external field} \]

\[ \epsilon'' \rightarrow \text{permittivity inside} \]

\[ \epsilon' \rightarrow \text{permittivity outside} \]

**Note:** typo in L6, L7

\[ \Phi' = -\vec{E} \cdot \vec{r} + \frac{A \epsilon' \cdot \vec{r}}{r^3} \]

\[ \Phi'' = -B \epsilon' \cdot \vec{r} \]

satisfies Laplace again

remains finite at center

depends only on \( \epsilon \)

boundary conditions at the surface

notice that \( \epsilon'' = B \epsilon' \) is uniform

continuity of potential gives

\[ -1 + A/R^3 = -B \]

\[ B = 1 - A/R^3 \]

\[ \epsilon'' = (1 - A/R^3) \epsilon' \]
continuity of normal component of induction

\[ \nabla \cdot \mathbf{D} = 0 \quad \text{in the absence of external charges} \]

gives

\[ D_z^{(i)} \bigg|_{z=R} = D_z^{(e)} \bigg|_{z=R} \quad \text{along the } z \text{- (pole-to-axis) axis} \]

\[ = \varepsilon^{(e)} E_z^{(e)} \bigg|_{z=R} \]

\[ = -\varepsilon^{(e)} \frac{\partial}{\partial z} \left( -\varepsilon z + A \frac{\varepsilon}{z^2} \right) \bigg|_{z=R} \]

\[ = -\varepsilon^{(e)} \left( -\varepsilon - 2 A \frac{\varepsilon}{z^3} \right) \bigg|_{z=R} \]

\[ = \varepsilon^{(i)} \varepsilon E \left( 1 + \frac{2A}{R^3} \right) \]

since \( D^{(i)} \) is uniform

\[ D^{(i)} = \varepsilon^{(e)} \varepsilon \left( 1 + \frac{2A}{R^3} \right) \]

\[ \varepsilon^{(i)} E^{(i)} \text{ in a dielectric} \]

\[ D_0 + \varepsilon^{(e)} E^{(e)} \]

\( \text{in a ferroelectric} \) (see 13.1).
\[ \varepsilon - \varepsilon^{(1)} = \frac{A \varepsilon}{R^3} \]

\[
\begin{align*}
D^{(1)} &= \varepsilon^{(0)} \varepsilon + 2 \varepsilon^{(0)} (\varepsilon - \varepsilon^{(1)}) \\
\left( (D^{(1)} + 2 \varepsilon^{(0)} \varepsilon^{(1)}) \right)^{\frac{1}{2}} &= \varepsilon^{(0)} \varepsilon
\end{align*}
\]

The field equation for static \( E : D \) are

\[ (6.1) \; \nabla \times E = 0 \; \nabla \cdot E = 4\pi \rho \; (6.2) \]

\[ 16.8 \; \nabla \cdot D = 4\pi \rho_{\text{ex}} \]

\[ \nabla \times D = \nabla \times E + 4\pi \nabla \times \rho \]

\[ = 4\pi \nabla \times \rho \quad \text{not necessarily zero} \]

In the absence of \( \rho_{\text{ex}} \) change,

\[ \varepsilon \varepsilon = \varepsilon + 4\pi \rho \Rightarrow P = \frac{\varepsilon - 1}{4\pi} \varepsilon = \kappa \varepsilon \]

\[ \therefore \nabla \times E = 0 \; \nabla \cdot E = 4\pi \rho \]

\[ \nabla \cdot D = 0 \; \nabla \times D = 4\pi \nabla \times \rho \]
The eqns for $H = B$ in a magnetizable ellipsoid are entirely analogous to those for $E = B$, respectively (see §30).

$$\nabla \times H = \frac{4\pi j}{c} = 0 \quad (j = \sigma E) \quad (58.4)$$

$$\nabla \cdot B = 0 \quad (58.3)$$

$$B = H + 4\pi M$$

\[\Rightarrow \]

$$\mu H \text{ (or } B_0 + \mu H \text{ for a semomagnetic medium)}$$

We can write $H = -\nabla \psi$ (analogously to $E = -\nabla \phi$) and find (by a derivation just like that of §8.1) that

$$\frac{1}{3} \left( B_{(s)}^{(s)} + 2 \mu^{(s)} H_{(s)}^{(s)} \right) = \mu^{(s)} \vec{P}$$

for a magnetizable sphere.

In the case of present interest, there is no external magnetic field ($E = 0$) and the sphere is surrounded by empty space ($\mu^{(e)} = 1$), so

$$B_{(s)}^{(s)} + 2 \mu^{(s)} H_{(s)}^{(s)} = 0$$
The two equations

\[ B + 2H = 0 \]

\[ B = H + 4\pi M \]

give

\[ B = \frac{8\pi M}{3} \]

In terms of the sphere's constant magnetization, then,

(from p. 1)

\[ \mathcal{E} = \frac{a^2}{2c} \mu_0 \frac{8\pi}{3} M \]

\[ \mathcal{E} = \frac{4\pi}{3} a^2 \mu_0 M = \frac{\mu_0 M}{ca} \]

\[ \frac{4\pi}{3} a^2 = \frac{4\pi}{3} a^3 / a = \frac{V_{\text{vol}}}{a} \text{ vol. of sphere} \]
Let's apply Eq. (8.1) (see p. 4) to the case of uniformly polarized (ferroelectric) sphere in the absence of an external field.

\[ D^{(i)} + 2\varepsilon^{(e)} E^{(i)} = 0 \]

\[ D^{(i)} = E^{(i)} + 4\pi P_0 \]

The actual constant polarization that is present when the ferroelectric sphere is immersed in (and has polarized) the surrounding dielectric medium.

This is where Amritavash asked his excellent question, "Doesn't \( D^{(i)} = \varepsilon^{(i)} E^{(i)} \) imply that \( (\varepsilon^{(i)} + 2\varepsilon^{(e)}) E^{(i)} = 0 \), so that one of the dielectric constants must be negative?"

The resolution is that in fact \( D^{(i)} = P_0 + \varepsilon^{(i)} E^{(i)} \) in the present situation, so things work out differently.

\[ E^{(i)} + 4\pi P_0 + 2\varepsilon^{(e)} E^{(i)} = 0 \]

\[ E^{(i)} = -\frac{4\pi}{1+2\varepsilon^{(e)}} P_0 \]
\[ D^{(c)} = E^{(c)} + 4\pi P_0 \]

\[ = 4\pi \left( 1 - \frac{1}{1 + 2\epsilon^{(c)}} \right) P_0 \]

\[ = 4\pi \left( \frac{2\epsilon^{(c)}}{1 + 2\epsilon^{(c)}} \right) P_0 \]

and \[ D_0 = D^{(c)} - \epsilon^{(c)} E^{(c)} \]

\[ = \left\{ \frac{8\pi}{1 + 2\epsilon^{(c)}} - \frac{4\pi \epsilon^{(c)}}{1 + 2\epsilon^{(c)}} \right\} P_0 \]

\[ D_0 = \frac{2 - \epsilon^{(c)}}{1 + 2\epsilon^{(c)}} \frac{4\pi}{P_0} \]

\[ \text{points in the same direction as } P_0 \]

\[ \text{E}^{(c)} \text{ points in the opposite direction} \]

\[ \text{D}_0 \text{ can point either way} \]

\[ \rightarrow P_0 \]

\[ \text{Notice that } E^{(e)} \text{ in the case drawn here is } \]

\[ \text{of } E^{(e)} = 1. \text{ Field lines change direction at}\]

\[ \text{surface} \]
unipolar induction: example of electromagnetic induction, mechanical energy being converted into electrical energy, through the medium of the magnetic field explained by forces acting on conduction electrons inside the magnet.

The vector $\mathbf{B}$ is symmetrical about the axis of a cylindrical magnet, and is unchanged when the magnet is set in rotation.

In the frame, no magnet is rotating if the external circuit is stationary, no electric fields can be induced in an electromagnetic sense.

Conduction electrons within the magnet take up orbital motion with net drift velocity

\[ \mathbf{v} = \omega \times \mathbf{r} \quad \text{(origin on axis)} \]

Conduction electrons experience a force $-e\mathbf{v} \times \mathbf{B}$ which is mainly directed toward the central axis.

- Negative charge appears in body of magnet
- Positive on its curved outer surface

In equilibrium

\[ F = -eE - e\mathbf{v} \times \mathbf{B} = 0 \]

\[ E = -\mathbf{v} \times \mathbf{B} \]
Aside: What are the lines of force for a vector field \( \mathbf{V}(\mathbf{r}) \)?

In 2D, how do we specify a curve corresponding, say, to \( \mathbf{F}(\mathbf{r}) = 0 \)?

\[
\mathbf{F}(\mathbf{r}) = 0 \quad \text{and} \quad \nabla \cdot \mathbf{F}(\mathbf{r}) = 0
\]

\[
\mathbf{d}l \times \frac{\mathbf{F}}{\partial x} + \mathbf{d}l \times \frac{\mathbf{F}}{\partial y} = 0
\]

\[
\frac{\partial x}{\partial y} = - \frac{\partial y}{\partial x}
\]

Suppose we have a vector field in 2D, what are its lines of force?

\[
\nabla \mathbf{V}(\mathbf{r}) = \mathbf{F}(\mathbf{r})
\]

what is \( \mathbf{V}(\mathbf{r}) \) tangent to the lines of force at every point

so is \( \nabla \mathbf{V}(\mathbf{r}) \)

\[
\mathbf{d}l = \mathbf{d}l \mathbf{\hat{v}}(\mathbf{r}) = \mathbf{d}l \frac{\mathbf{V}(\mathbf{r})}{\mathbf{V}(\mathbf{r})}
\]

\[
\mathbf{V}(\mathbf{r}) \mathbf{d}l = \mathbf{d}l \mathbf{v}(\mathbf{r})
\]

\[
\mathbf{d}l = \mathbf{c}(\mathbf{c} + \mathbf{d}l) - \mathbf{c}(\mathbf{c})
\]

\[
\mathbf{V}(\mathbf{r}) \mathbf{c}(\mathbf{c} + \mathbf{d}l) = \mathbf{V}(\mathbf{r}) \mathbf{c}(\mathbf{c}) + \mathbf{d}l \mathbf{v}(\mathbf{r})
\]

LINES OF FORCE
If \( \mathbf{E} \) is to specify a line of force for the vector field \( \mathbf{V} \), then

\[
\frac{d\mathbf{E}}{d\ell} = \mathbf{V}(\ell)
\]

Back to Montgomery

potential \( \mathbf{V} \) corresponding to \( \mathbf{E} = -\mathbf{x} \times \mathbf{B} \)

field is zero along axis of cylinder (where we take \( \mathbf{V}(x=0,y=0,z) = 0 \)

For pt. \( P \) on surface of cylinder

\[
\mathbf{V}_P = -\int_0^r s \mathbf{E} \cdot ds = \int_0^r \mathbf{V} \times \mathbf{B} \cdot ds
\]

\[
= \int_0^r dp \left( \mathbf{V} \times \mathbf{B} \right)_p = \int_0^r dp \left( v_y B_z - v_z B_y \right)
\]

\[
\mathbf{V}_P = \int_0^r dp \frac{v_y}{r} B_z
\]

\[
\mathbf{V}_P = \frac{w}{2\pi} \Phi_B
\]

\[

\mathbf{V}_P = \frac{w}{2\pi} \int_0^r dp \frac{2\pi p B_z}{r} = \frac{w}{2\pi} \Phi_B
\]

\[
(\nabla \times \mathbf{E})_i = (D \times (\nabla \times \mathbf{B}))_i = \varepsilon_{ijk} \varepsilon_{kln} \frac{\partial}{\partial r_j} v_k B_l
\]

\[
= (\delta_{ij} \delta_{jm} - \delta_{im} \delta_{j} \delta_{n}) \frac{\partial}{\partial r_j} v_k B_l
\]

\[
= \frac{\partial}{\partial r_j} v_j B_i - \frac{\partial}{\partial r_j} v_i B_j
\]
\[ (\nabla \times \mathbf{E})_i = \frac{2}{\partial_j} \varepsilon_{ilm} \nabla \times \mathbf{E}_j - \frac{2}{\partial_j} \varepsilon_{ilm} \nabla \cdot \mathbf{E}_j \]

\[ = \nabla \left( \varepsilon_{ilm} \frac{2}{\partial_j} \nabla \mathbf{E}_j - \varepsilon_{ilm} \frac{2}{\partial_j} \nabla \cdot \mathbf{E}_j \right) \]

\[ = \nabla \left( \varepsilon_{ilm} \left( \delta_j \mathbf{E}_j + \nabla \frac{\partial B_i}{\partial j} \right) - \varepsilon_{ilm} \left( \delta_j \mathbf{E}_j + \nabla \frac{\partial B_i}{\partial j} \right) \right) \]

\[ = \nabla \left( \varepsilon_{ilm} B_m - \varepsilon_{ilm} \nabla \frac{\partial B_i}{\partial j} \right) \]

\[ (\nabla \times \mathbf{E})_i = (\mathbf{\nabla} \times \mathbf{B})_i - (\mathbf{\nabla} \times \mathbf{E}) \cdot \nabla B_i \]

4. This is supposed to be zero?

TRY AGAIN

\[ (\nabla \times \mathbf{E})_i = -(\nabla \times (\mathbf{\nabla} \times \mathbf{B}))_i = (\nabla \times ((\mathbf{\nabla} \times \mathbf{E}) \times \mathbf{B}))_i \]

\[ = \varepsilon_{ijk} \frac{\partial}{\partial j} \left( \left( \mathbf{\nabla} \times \mathbf{E} \right)_k \right) \]

\[ = \varepsilon_{ijk} \frac{\partial}{\partial j} \varepsilon_{elm} \left( \mathbf{\nabla} \times \mathbf{E} \right)_l \mathbf{B}_m \]

\[ = \varepsilon_{kij} \varepsilon_{elm} \frac{\partial}{\partial j} \epsilon_{pq} \epsilon_{r} \mathbf{B}_m \]

\[ = \left( \delta_{ikj} \delta_{lm} - \delta_{il} \delta_{jm} \right) \epsilon_{pq} \epsilon_{r} \frac{\partial}{\partial j} \mathbf{B}_m \]

\[ = \left( \delta_{ikj} \delta_{lm} - \delta_{il} \delta_{jm} \right) \epsilon_{pq} \epsilon_{r} \left( \delta_{pq} \mathbf{B}_m + \mathbf{B}_m \frac{\partial \mathbf{B}_m}{\partial j} \right) \]
\((\nabla \times E)_i = \varepsilon_{ijk} \psi_{mp} \delta_{qB} B_j + \varepsilon_{ijk} \psi_{mp} \frac{\partial B_i}{\partial r_j} - \varepsilon_{ijk} \psi_{mp} \delta_{qB} B_i - \varepsilon_{ijk} \psi_{mp} \frac{\partial B_i}{\partial r_j}\)

\((\nabla \times E)_i = \varepsilon_{ipj} \psi_{mp} B_j - \varepsilon_{ipj} \psi_{mp} \frac{\partial B_i}{\partial r_j}\)

\(= (\mathbf{w} \times \mathbf{B})_j - (\mathbf{w} \times \mathbf{E})_j \frac{\partial}{\partial r_j} B_i\)

\(\nabla \times E = \mathbf{w} \times \mathbf{B} - \mathbf{v} \cdot \nabla \mathbf{B}\)

- \(\mathbf{w} \times \mathbf{B}\)

- \(\mathbf{v} \cdot \nabla \mathbf{B}\)

\(\mathbf{v} \frac{\partial \mathbf{B}}{\partial y}\)

\(\nabla \cdot \mathbf{E} = 0\)

same as before

why should this be zero?

IT IS, see p. 6

due to cylindrical symmetry

has component in negative \(\phi\) direction
\[ \nabla \times E = \omega \times B - (\omega \times \mathbf{e}) \cdot \nabla \mathbf{B} \]

\[ \omega = \omega \hat{e}_z \]
\[ \mathbf{e} = \hat{p} \mathbf{p} + \hat{z} \mathbf{z} \]
\[ \omega \times \mathbf{e} = \omega \hat{p} \mathbf{p} \]
\[ \hat{p} \cdot \nabla = \frac{1}{\rho} \frac{\partial}{\partial \rho} \]

\[ \mathbf{B} = B_\rho \hat{p} + B_\phi \hat{\phi} \]
\[ \omega \times \mathbf{B} = \omega B_\rho \hat{\phi} \]

\[ \nabla \times \mathbf{E} = \omega B_\rho \hat{\phi} - \omega \frac{\partial}{\partial \phi} \mathbf{B} \]
\[ \nabla \times \mathbf{E} = \omega \left[ B_\rho \hat{\phi} - \frac{\partial}{\partial \phi} \left( B_\rho \hat{p} + B_\phi \hat{\phi} \right) \right] \]
\[ = \omega B_\rho \left( \hat{\phi} - \frac{\partial \hat{p}}{\partial \phi} \right) \]

\[ \hat{p} = \hat{x} \cos \phi + \hat{y} \sin \phi \]
\[ \hat{\phi} = \hat{y} \cos \phi - \hat{x} \sin \phi \]
\[ \frac{\partial \hat{p}}{\partial \phi} = -\hat{x} \sin \phi + \hat{y} \cos \phi = \hat{\phi} \]

\[ \nabla \times \mathbf{E} = \omega B_\rho (\hat{\phi} - \hat{\phi}) = 0 \]

As stated as calculated.
unipolar generator: \( I = w \times z \) does not represent Drift velocity when current is being drawn from DC system.
EMF and mechanical energy converted into electrical energy per coulomb of charge circulated.

Torque acting on magnet:

\[
\tau = J \times B
\]

where \( J \) is current density inside magnet, density of conduction electrons.

Torque = Force \times Distance = \( \vec{F} \times \vec{r} \)

\[
S = \int \text{d}V \times (\vec{E} \times \vec{B})
\]

Torque excited by electrons on magnet.

\[
\text{d}S = \rho d\vec{g} d\vec{d} \text{d}V_n
\]

\( J' \) - projection of \( J \) onto \( p^2 \)-plane

Axial component

\[
G_z = \int \text{d}V \hat{z} \cdot (\vec{J} (\vec{E} \cdot \vec{B}) - \vec{B} (\vec{E} \cdot \vec{J}))
\]

\[
= \int \text{d}V \left( J^2 (\vec{E} \cdot \vec{B}) - B_z (\vec{E} \cdot \vec{J}) \right)
\]

\[
= \int \text{d}V \left( J^2 \rho \vec{B} + \vec{J} \times \vec{B} \right) - B_z (\vec{J} \times \vec{B} - \vec{J} \rho \vec{B})
\]

\[
= \int \text{d}V \rho (J^2 \vec{B} - B_z \vec{J}) = \int \text{d}V \rho (J^2 \vec{B} - B_z \vec{J})
\]
\[ G_z = - \int d\Omega \rho J' B \sin \alpha \]

\[ = - \int \delta g \, d\ell \, d\Omega_B \rho^2 J' B \sin \alpha \]

\[ \text{angle between } J' \frac{g}{B} \]

\[ \text{J}_\phi, \text{the tangential component of } J \text{ does not contribute to } G_z \]

\[ \left( \hat{z} \times \hat{p} \right) \times \left( \hat{r} J_\phi \times \left( \hat{p} B_p + \hat{z} B_z \right) \right) = \left( \hat{z} \times \hat{p} \right) J_\phi \left( -\hat{z} B_p + \hat{p} B_z \right) \]

\[ = J_\phi \left( \hat{z} z B_z + \hat{p} p B_p \right) \]

\[ = \hat{p} J_\phi \left( z B_z + p B_p \right) \]

\[ \text{points in } \hat{p} \text{ direction;} \]

\[ \text{no } z \text{ component.} \]

\[ \text{flux enclosed by wall of sheath} \]

\[ d\Omega_B = 2\pi \rho B \, d\varpi \]

\[ \rho B d\varpi = \frac{d\Omega_B}{2\pi} \]

\[ G_z = - \int \delta g \, d\ell \, d\Omega_B \frac{p}{2\pi} J' \sin \alpha \quad (11) \]

The current \( I \) enters at \( A \) and exits at \( B \) and equals the total current across the wall of the sheath.

\[ I = \int \rho d\varphi \sin \alpha \, d\ell \]

\[ G_z = - \frac{I}{2\pi} \int d\Omega_B = - \frac{\Phi_B}{2\pi} I \quad (13) \]

\[ = - \Gamma \]
electrical react a torque $-P$ on the magnet; to maintain steady rotation therefore requires the application of a mechanical torque to the magnet.

The mechanical power supplied is $P_w$ and this equals $EI$:

$$P_w = \frac{Iw}{2\pi} B = EI$$

$$\therefore \quad E = \frac{w}{2\pi} B$$

which equals the open circuit voltage between $A$ and $P$.

Current density is complicated but constant, no current filaments are cutting thru stationary lines of force, EM induction w both $J$ & $B$ constant.

![Diagram of motor or generator with current $I$, armature $A$, and field $F$]

engine exerts a torque $C > 0$ on armature.

magnet exerts torque $-C$ on armature.

In the unipolar motor, the engine exerts torque $+P$ on the magnet.

no action and reaction between magnet and armature because they are the same body.

$$g = \frac{1}{2} E \times H \quad \text{linear momentum of field}$$

$$L = \int dv \; \mathbf{r} \times \mathbf{g} \quad \text{in the unipolar generator/motor both } E \times H \text{ are constant, so mechanical } \text{ EM angular momentum are both conserved.}$$
The existence of the field momentum means that we cannot always divide the system into mechanical parts which are interacting w/ ea. Then in a chin-reach pairs total torque on framework must be zero.

\[
E = -v \times B
\]

\[
g \alpha \mathbf{E} \times \mathbf{B}
\]

\[
L = \int \mathbf{E} \times \mathbf{g} \cdot \mathbf{\hat{z}} \quad \text{along axis of magnet}
\]

Appendix

**E** B in gap same as B in magnet

\[
\text{WHY'S THIS? Because } \nabla \cdot \mathbf{B} = 0:
\]

\[
o = \oint \mathbf{D} \cdot \mathbf{B} \, d\mathbf{V} = \oint \mathbf{B} \cdot d\mathbf{S}
\]

\[
= B_{z}^{\text{top}} A - B_{z}^{\text{bottom}} A
\]

\[
\therefore \quad B_{z}^{\text{bottom}} = B_{z}^{\text{top}}
\]

B in the gap equals B in the magnet.
Battery $E_0$ maintains constant current $I$ in the circuit.
no need to rotate magnetic.
total axial component of torque on circuit is $G_0$.

$$G_0 = G_{op} + G_{pe}$$

virtual $\delta \phi$ of circuit, no emf induced because flux linking

circuit is unchanged. I remains constant.

work done on the circuit $\delta W = -G_0 \delta \phi$

energy doesn't change due to cylindrical symmetry

$$0 = G_0 = G_{op} + G_{pe} \implies G_{pe} = -G_{op}$$


$$G_{op} = \int_a^b dV \left( \mathbf{e} \times \left( \mathbf{e} \times \mathbf{B} \right) \right)_z$$

$$= \int_a^b dV \left\{ f_x (e_y e_z B_z) - f_y (e_x e_z B_z) \right\}$$

$$= \int_a^b dV \rho (e_x e_z B_z) = \int_a^b dV \rho (e_v e_z B_z - e_v e_z B_z)$$

$$= -e \int_a^b dV \rho v_x B_z = - \int_a^b d\rho \frac{I}{e} \frac{B_z}{\pi}$$

$$G_{op} = -\frac{I}{2\pi} \int_a^b \frac{2\pi \rho B_z}{\pi} = -\frac{I}{2\pi} \Phi_B = -\Gamma$$

$$\therefore G_{pe} = -\Gamma$$
Unipolar induction: a neglected topic in the teaching of electromagnetism

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Abstract. When a cylindrical magnet is rotated about its axis, electric fields develop on its surface which can be used to generate continuous currents. This effect is an example of electromagnetic induction, mechanical energy being converted into electrical energy through the mediation of the magnetic field. It is suggested that the effect can be explained most simply in terms of the forces acting on conduction electrons inside the magnet, rather than in terms of flux linkage and the cutting of lines of force.

1. Introduction

In 1831, shortly after his discovery of electromagnetic induction, Michael Faraday carried out the experiment shown schematically in figure 1. A cylindrical steel magnet was hung vertically with its lower pole immersed in a bottle of mercury, and its upper pole connected to the circuit shown. When the magnet was made to rotate about its axis, a continuous current was observed in the galvanometer [1, p 204]. This should be compared with an earlier experiment performed by Ampère in 1821, if the galvanometer is replaced by a Voltaic pile so that a current is driven round the circuit, the magnet is found to rotate spontaneously about its own axis [1, p 168].

![Figure 1. Diagram of Faraday’s experiment of 1831.](image-url)
close connection between these two experiments was not well understood at the time, because the principle of conservation and transformation of energy had not yet been established.)

Faraday’s discovery sparked off a debate among physicists and electrical engineers which lasted the rest of the nineteenth century; this has been fully documented in a fine review by Miller [2]. In 1841 Weber christened the effect unipolar induction, because he believed that only one of the poles of the magnet was involved; he extended the term to include more general situations, in which a disc or hollow cylinder of copper rotated about the axis of the magnet. Attempts were made to explain unipolar induction in terms of circuit elements cutting through magnetic lines of force, but these explanations raised an immediate question. When a magnet rotates, do its lines of force rotate with it, and if so do they create an electromotive force as they pass through a stationary conductor?

Faraday himself believed that as the magnet rotates its lines of force remain stationary, and this brought him into conflict with Ampère’s theory, in which magnetic properties arise from current loops within individual molecules in the body of the magnet, so that if lines of force exist they should be carried round with the magnet as it rotates. Physicists found themselves divided into roughly three camps: those who believed that the lines of force rotated, those who believed that they did not, and those who believed that lines of force were merely a representation of the field, so that the question had no physical meaning. It turned out to be very difficult to devise an experiment which could decide unambiguously between these hypotheses, and in the nineteenth century the theory of electromagnetism was not sufficiently established for it to be able to decide the matter on theoretical grounds alone. It was only in the twentieth century, with the general acceptance of Maxwell’s equations, the electron theory of Lorentz and the principles of relativity, that a consensus on unipolar induction could emerge.

Although there was much confusion in the early years about the theory of unipolar induction, it was accepted that the effect actually exists, and there were some attempts to exploit it commercially. In 1912 the Westinghouse Corporation built a colossal unipolar generator which delivered a direct current of 7700 A at a voltage of 264 V. However, such machines were never put into general production, because it was conceded that AC generators provided a better technology [2].

Students today still find unipolar induction a puzzling phenomenon, partly because the magnetic flux linking the circuit is constant in time. (I confess that when I first read an account of Faraday’s experiment I was convinced that it was wrong, and I have found similar reactions among other teachers.) In the following sections I will describe the theory of unipolar induction in the simplest possible way, without invoking relativity, and then I will consider the impact of these ideas on the teaching of electromagnetism in general.

2. The origin of the electromotive force

The fact that one can move or rotate a magnet does not always mean that one can move or rotate its magnetic field. The field is characterized by the value of the magnetic vector $B$ at every point in space; if all of these vectors are independent of time, the field is constant, regardless of whether the magnet producing the field is moving or not. In figure 1 the magnetic field is symmetrical about the cylindrical axis, and is unchanged when the magnet is set in rotation. Magnetic lines of force are a useful description of the magnetic field, but they are not real in themselves.

We shall stick to the laboratory frame of reference, in which the magnet is rotating and the external circuit is stationary. Within this frame the magnetic field is constant, and no electric fields can be induced in the electromagnetic sense.

Consider first the case of a cylindrical magnet which is rotating about its axis in complete electrical isolation (see figure 2). Let the radius of the magnet be $a$, and its angular velocity $\omega$. Conduction electrons within the magnet undergo collisions with the moving atoms, and this causes them to take up the rotational motion. At each point the electrons have a net drift

\[ \mathbf{D} \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \]
velocity \( v \), given by
\[
\vec{v} = \vec{\omega} \times \vec{r}
\]
where \( \vec{r} \) is the position vector of the point in question, relative to an origin which is chosen to lie on the magnet axis.

The conduction electrons experience a force \( -e \vec{v} \times \vec{B} \) which is mainly directed towards the central axis; as a result a negative charge appears in the body of the magnet, and a positive charge on its curved outer surface. In equilibrium an electrostatic field is set up, such that the total Lorentz force on the conduction electrons is zero:
\[
\vec{F} = -e \vec{E} - e \vec{v} \times \vec{B} = 0.
\]
Therefore
\[
\vec{E} = -\vec{v} \times \vec{B}.
\]  

We now calculate the potential \( V \) corresponding to this electric field. Equation (3) shows that along the axis \( E \) is zero, and we can take \( V \) as zero at all points along this axis. Hence, at a general point \( P \) on the surface of the magnet
\[
V_P = -\int_0^p \vec{E} \cdot d\vec{s} = +\int_0^p \vec{v} \times \vec{B} \cdot d\vec{s}.
\]
Expressing this integral in terms of cylindrical polar coordinates \( (\rho, \phi, z) \)
\[
V_P = -\int_0^\rho \int_0^\phi \int_0^z \omega \vec{B} \cdot d\rho \cdot d\phi \cdot dz = \frac{\omega}{2\pi} \int_0^\rho \int_0^\phi \int_0^z \vec{B} \cdot 2\pi \rho \cdot d\rho \cdot d\phi \cdot dz = \frac{\omega}{2\pi} \Phi_B
\]
where \( \Phi_B \) is the total magnetic flux passing through the horizontal circle \( QP \).

This argument can be extended to any point inside the magnet; for example, at point \( S \) the potential \( V \) is given by equation (5), provided that \( \Phi_B \) is interpreted as the magnetic flux passing through the circle \( TS \). It follows that the equipotential surfaces are obtained by rotating the magnetic lines of force about the axis of the cylinder, and one can also confirm that \( E \) is irrotational and is described completely by a scalar potential \( V \).
Now consider the situation in figure 3, where brush contacts have been attached to points A and B, and they are connected by a stationary wire circuit of resistance $R$. There is a reduction in the voltage $V$, and inside the magnet the electric force becomes weaker than the magnetic force, hence electrons are drawn in towards the axis and a conventional current $i$ circulates in the sense shown in figure 3.

In order to maintain current indefinitely there has to be a continuous input of energy, and we now need to see where this energy is coming from. The lines of current flow inside the magnet are complicated, but fortunately we shall not have to calculate them in detail. In figure 2 the flow lines are circles, but in figure 3 they form inward spirals, and this introduces a new component into the magnetic force.

It is clear from figure 4 that if the drift velocity $v$ has a component in the negative $\rho$ direction, the magnetic force $-ev \times B$ must have a component in the negative $\phi$ direction, and this is directed against the angular velocity $\omega$. By means of their collisions the electrons exert a negative torque on the magnet, and to maintain its rotation a positive torque has to be supplied mechanically from outside. (We shall regard a torque as positive if it acts in the same sense as $\omega$.) Hence the mechanical work performed on the magnet is converted into electrical energy in the external circuit. Note that no energy is given to the electrons by the magnetic field itself, because the magnetic force $-ev \times B$ is at right angles to the velocity $v$. The electrons acquire energy and net momentum from the collisions, and the role of the magnetic field is to redirect this momentum towards the axis, so that a current flows round the circuit.

(Although we have concentrated on the unipolar generator, it is clear that the same system can act as a unipolar electric motor as in Ampère's experiment [1, p 168]. If we replace the load resistor $R$ in figure 3 by a power source providing a current $i$ in the same direction, the magnet will rotate in an anticlockwise sense about the $z$ axis, and will deliver a torque to an external mechanical system.)

3. The calculation of the electromotive force

We now return to the case of the unipolar generator. It is clear from figure 4 that equation (1) does not represent the drift velocity when current is being drawn from the system, and it does not even describe its tangential component $v_{\phi}$. We must be careful not to use equation (1) when the EMF is being calculated.

The EMF can be defined as the amount of mechanical energy converted into electrical energy per coulomb of charge circulated, and we begin by calculating the mechanical torque acting on the magnet. This can be done using the following argument due to Page and Adams [3]. The current density inside the magnet is given by

$$J = -nev$$

(6)
where \( n \) is the density of the conduction electrons.

The torque exerted by the electrons on the magnet is given by

\[
G = \iiint \mathbf{r} \times (\mathbf{J} \times \mathbf{B}) \, d\tau
\]

integrated over the volume of the magnet.

In figure 5 we choose a particular form for the volume element \( d\tau \). Take two neighbouring lines of force, and rotate them round the cylinder axis to form a curved sheath of magnetic flux. At the point \( Z \), \( d\tau \) is subtended between the inner and outer surfaces:

\[
d\tau = \rho \, d\phi \, dl \, d\sigma
\]

Let \( J' \) be the projection of \( J \) onto the \( \{\rho, z\} \) plane. (\( J' \) is \( J \) without its tangential component \( J_{\theta} \).) The axial component of \( G \) has the form

\[
G_z = - \iiint \rho B J' \sin \alpha \, d\phi \, dl \, d\sigma
\]

where \( \alpha \) is the angle between \( B \) and \( J' \). (Note that \( J_{\theta} \), the tangential component of \( J \), does not contribute to \( G_z \); see figure 4.)

The flux enclosed in the walls of the sheath is given by

\[
d\Phi_B = 2\pi \rho B \, d\sigma
\]

therefore

\[
G_z = - \iiint \frac{\rho}{2\pi} J' \sin \alpha \, d\phi \, dl \, d\Phi_B
\]

The current enters at \( A \) and leaves at \( P \), and it equals the total current across the walls of the sheath:

\[
I = \iint \rho J' \sin \alpha \, d\phi \, dl
\]

therefore

\[
G_z = -\frac{I}{2\pi} \int d\Phi_B = -\frac{I}{2\pi} \Phi_B
\]
where $\Phi_B$ is the flux passing through the circle QP.

Let

$$\Gamma = \frac{I}{2\pi} \Phi_B.$$  \hspace{1cm} (14)

Hence the conduction electrons exert a torque $-\Gamma$ on the magnet, and it follows that the mechanical torque on the magnet is $+\Gamma$.

It is now easy to calculate the EMF $\mathcal{E}$. The mechanical power supplied is $\Gamma \omega$ and this is equal to $\mathcal{E} I$:

$$\Gamma \omega = \frac{I \omega}{2\pi} \Phi_B = \mathcal{E} I$$  \hspace{1cm} (15)

therefore

$$\mathcal{E} = \frac{\omega}{2\pi} \Phi_B.$$  \hspace{1cm} (16)

(It is reassuring to see that $\mathcal{E}$ equals the voltage between A and P when the system is on open circuit—see equation (5).)

The current density $J$ has a complicated structure inside the magnet, but it is important to realize that it is constant in time. No current filaments are cutting through stationary lines of force. Here we have a case of electromagnetic induction in which both the magnetic field $B$ and the current density $J$ are constant.

Before leaving the calculation of the EMF, we should compare the results we have obtained with the corresponding results for a relativistic theory. The basic assumptions of the relativistic treatment are of course different, but in the laboratory frame of reference the conclusions are very similar, at least to first order in $a \omega/c$ where $a$ is the radius of the magnet. Equation (3) is still valid for the electric field $E$ inside an isolated rotating magnet. However, the discussion which this electric field is generated by a displacement of conduction electrons, and this is not entirely true in the relativistic theory: $E$ arises partly from the displacement of the conduction electrons, and partly from an electric polarization $P$ of the medium. This subject falls outside the scope of the present paper, and a good account of it is given by Rosser [4].

4. The conservation of angular momentum

Figure 6 shows a conventional DC generator, which should be compared with the unipolar generator in figure 7. (Note that in the conventional generator the axis of rotation is at right angles to the magnetic field, in the unipolar generator it is parallel to the field.) In figure 6 there is a source of mechanical power or engine, which is connected by a shaft to the armature. Suppose that the engine exerts a torque $+\Gamma$ on the armature, where $C$ is a positive quantity. When the system is running at constant speed the magnet exerts a torque $-\Gamma$ on the armature, and the armature reacts against the magnet and exerts a torque $+C$ upon it. When the system is working backwards as an electric motor, it is the reaction of the armature against the magnet which provides the mechanical power.

In figure 7 the engine exerts a torque $+\Gamma$ on the magnet, but there is no action and reaction between the magnet and armature, because they have become the same body. When the system is working backwards as an electric motor, one has the uncomfortable feeling that the magnet is pulling itself round by its own bootstraps [5].

This paradox illustrates the point that one has to be extremely careful when one applies Newton's third law to electromagnetic systems ([7], [8, section 27-9]). The electromagnetic field itself possesses momentum, a concept which is more familiar to most of us in the case of photons than it is for static fields. The density of linear momentum is given by Poynting's vector, divided by $c^3$:

$$g = \frac{1}{c^3} E \times H.$$  \hspace{1cm} (17)

† There is a comment on this paper by Scadon and Hemriksen [6].
Hence the angular momentum of the field is given by

\[ L = \iiint r \times g \, d\tau \]  \hspace{1cm} (18)

and this has to be added to the angular momentum of the mechanical parts of the system, to give the total angular momentum which is conserved. In all the cases we have considered both \( E \) and \( H \) are time-independent, and it follows that the electromagnetic and the mechanical angular momenta are both conserved individually. However, the existence of the field momentum implies that we cannot always divide the system into mechanical parts which are interacting with each other in action–reaction pairs. But if the complete system is mounted on a freestanding framework (shown shaded in the diagrams), then we can say that the total torque on this framework must be zero.

These considerations are important in the unipolar generator, because one can see from figure 2 that the field has an angular momentum directed along the axis of rotation, given by equation (18). In figure 7 the engine exerts a torque \( +\Gamma \) on the magnet, so that it exerts a torque \( -\Gamma \) on the framework. If the external circuit were to rotate with the magnet the EMF would be zero; for the generator to work the external circuit must be attached to the frame. It is shown in the appendix that the magnetic field exerts a torque on the external circuit, and the magnitude of this torque is \( +\Gamma \). This is transmitted to the frame, and it cancels the torque on the frame exerted by the engine.

In the conventional generator the magnetic field is at right angles to the axis of rotation, and it seems that the field momentum can be neglected. In figure 6 the torque \( -C \) which the engine exerts on the frame is balanced by the torque \( +C \) which is transmitted through the magnet, so that the total torque on the frame is again zero. One might argue that the external circuit could experience a small torque in the fringing field of the magnet, and that in order to prevent the external circuit from rotating it has to be attached to the frame. However, this is not important, because any torque exerted by the external circuit on the frame would be cancelled by its torque on the magnet.
5. Discussion

Suppose we have a closed loop of thin wire which is moving or changing shape in a magnetic field which is constant in time. Standard arguments show that an EMF is induced in the wire, given by the equation

\[ \mathcal{E} = \oint v \times B \cdot ds = -\frac{d}{dt} \Phi_B \]  \hspace{1cm} (19)

where \( ds \) is a line element of the loop which is moving at velocity \( v \), and \( \Phi_B \) is the magnetic flux linking the loop.

Equation (19) is immensely useful, and it leads on naturally to the discussion of time-varying fields, Maxwell’s equations and relativity. On the other hand, it can be misrepresented as the basic principle of electromagnetic induction, linking it inevitably with flux changes and the cutting of lines of force. The student can be forgiven for thinking:

“If the magnetic field \( B \) is constant, and the current density \( J \) is also constant, no EMF can be induced.”

This is an entirely false conclusion, as the counterexample of the unipolar generator shows. In the case of the moving wire, the current at any point is confined by the direction and velocity of the line element \( ds \). In the case of a continuous medium such as the magnet the current is not confined in this way, and there is no valid argument which takes us from the particular to the general case. It is not clear what meaning can be attached to equation (19) in the case of a unipolar generator which is delivering current to an external load.

A very clear but brief exposition of this argument is to be found in Feynman’s Lectures [8, section 17-2]. He discusses a slightly different form of unipolar generator known as Faraday’s disc, another of Faraday’s discoveries in his great year of 1831 [1, p 196]. In this system the current density \( J \) is constant in time, just as it is for the rotating magnet, and Feynman stresses that equation (19) is not a suitable starting point for the discussion. His conclusion is that when we are studying electromagnetic induction, the correct physics can always be obtained by considering Maxwell’s equations and the Lorentz forces on the electrons.

In these days of crowded syllabuses and examination deadlines, there are good arguments for omitting minor topics such as unipolar induction. However, I suggest that the material presented to students should be designed in such a way that the unipolar generator is not completely incomprehensible should they happen to come across it. It is better to speak of a magnetic field which is changing rather than moving or rotating, and lines of force should be presented as a good device for describing the field, but not as something real in themselves.

Most of all, I would suggest that the electromotive force should be defined in terms of energy transfer, and not in terms of a line integral as in equation (19). The EMF of a chemical cell is the amount of chemical energy converted into electrical energy per coulomb circulated; the same definition works excellently for an electromagnetic generator, provided that one substitutes mechanical energy for chemical energy.

Acknowledgment

I am grateful to Mr Stuart Leadstone for several lively discussions on this topic.

Appendix. The torque acting on the external circuit

Suppose that the magnet in figure 2 has been cut into two cylindrical sections \( K \) and \( L \), and that a narrow hole \( AO \) has been drilled down the axis of section \( K \), as indicated in figure A1. A rigid insulated wire circuit \( OCDE \) is inserted as shown, and sections \( K \) and \( L \) are brought close together, so that \( B_z \), the vertical component of the field in the gap, is equal to \( B_z \) inside.
the magnet. A battery $E_b$ maintains a constant current $I$ in the circuit. There is no need to rotate the magnet.

The field of the magnet exerts forces on the various parts of the circuit, and let the total axial component of the torque on the circuit be $G_O$. We can think of this torque as the sum of two terms; the torque $G_{OP}$ acting on the section $OP$, and the torque $G_{PE}$ acting on the rest of the circuit:

$$G_O = G_{OP} + G_{PE}. \quad (A1)$$

Now make a virtual rotation $\delta \phi$ of the circuit about the axis of the magnet. No EMF is induced because the flux linking the circuit is unchanged, and the current $I$ remains constant. The work performed on the circuit is

$$\delta W = -G_O \delta \phi. \quad (A2)$$

It is clear from the cylindrical symmetry that the total energy of the system is unchanged, and in fact no work is performed in the rotation:

$$\delta W = -G_O \delta \phi = 0$$

therefore

$$G_O = G_{OP} + G_{PE} = 0 \quad (A3)$$

so that the total torque on the circuit is zero. $G_{OP}$ is easily calculated:

$$G_{OP} = -\int_0^a \rho B_z I \, d\rho = -\frac{I}{2\pi} \int_0^a 2\pi \rho B_z \, d\rho = -\frac{I}{2\pi} \Phi_B$$

$$= -\Gamma \quad (A4)$$

where $\Gamma$ is defined in equation (14).

Hence from equation (A3) we have

$$G_{PE} = +\Gamma. \quad (A5)$$
This is a very general result, and it does not depend on the shape or dimensions of the external circuit. Note that the system in figure A1 could not operate as a unipolar electric motor. To get unipolar action we would have to make a mechanical break at point P, in such a way that the two parts of the circuit could move independently while maintaining electrical contact. For example, if a thin copper disc were placed in the gap between K and L, with brush contacts to the external circuit, it would rotate as a Faraday disc.

References

Excitations of currents by acceleration

discussion of motion of a conductor in §6.3 neglected effects of acceleration.
accelerator motion of a metallic is equivalent to producing an additional inertia force on the conduction electrons.

\[ \ddot{v} = \text{acceleration of the conductor} \]
\[ m = \text{mass of electron} \]
\[ -e = \text{charge of the electron} \]

"Force" on electron is \(-e\ddot{v}\) affects the electron in the same way as an electric field.

\[
\frac{-e\ddot{v}}{e} = \frac{m\ddot{v}}{e}
\]

Effective electric field on the conduction electrons in an accelerated metal is

\[ E' = E + \frac{m\ddot{v}}{e} \]

Current density

\[ j = \sigma E' = \sigma (E + \frac{m\ddot{v}}{e}) \]

\[ E = E' - \frac{m\ddot{v}}{e} \]

\[ \nabla \times E = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \]

\[ \nabla \times E' = \frac{m}{e} \nabla \times \ddot{v} - \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \]

\[ \mathbf{v} = \mathbf{u} + \mathbf{\omega} \times \mathbf{r} \]

angular velocity

translational velocity
\[ \dot{\mathbf{y}} = \ddot{\mathbf{u}} + \omega \times \mathbf{v} + \dot{\omega} \times \mathbf{r} = \ddot{\mathbf{y}} + \omega \times (\dot{\mathbf{r}} + \dot{\omega} \times \mathbf{r}) + \dot{\omega} \times \mathbf{r} \]

\[ = \ddot{\mathbf{y}} + \omega \times \mathbf{v} + \omega (\dot{\omega} \times \mathbf{r}) - \nabla (\omega \cdot \dot{\mathbf{r}}) \mathbf{r}^2 + \omega \times \dot{\omega} \times \mathbf{r} \]

\[ \text{indep of } \mathbf{r} \]

\[ \therefore \nabla \times \dot{\mathbf{y}} = -\omega \times (\mathbf{r} \times \dot{\mathbf{r}}) - (\nabla \times \dot{\mathbf{r}}) \mathbf{r}^2 + \nabla \times (\dot{\omega} \times \mathbf{r}) \]

\[ = -\frac{\mathbf{r} \times \dot{\mathbf{r}}}{\mathbf{r}^2} + \nabla \times (\dot{\omega} \times \mathbf{r}) \]

\[ = \dot{\mathbf{r}} (\dot{\omega} \times \mathbf{r}) - (\dot{\omega} \cdot \mathbf{r}) \mathbf{r} \]

\[ \nabla \times \dot{\mathbf{y}} = 3 \dot{\mathbf{r}} - \dot{\mathbf{r}} = 2 \dot{\mathbf{r}} \]

\[ \nabla \times \mathbf{E}' = -\frac{1}{c} \frac{\partial \mathbf{H}'}{\partial t} + \frac{\mathbf{m} \mathbf{r}}{c^2} \]

\[ \nabla \times \mathbf{E}' = -\frac{1}{c} \frac{\partial \mathbf{H}'}{\partial t} \]

\[ \mathbf{H}' = \mathbf{H} - \frac{2 \mathbf{m} \mathbf{c}}{e} \frac{\mathbf{r}}{c^2} \]

since \( \mathbf{r} \) is indep. of coords

\[ \nabla \times \mathbf{H}' = \nabla \times \mathbf{H} = \frac{4 \pi}{c} \mathbf{J} = \frac{4 \pi}{c} \sigma \mathbf{E}' \]

\[ \nabla \times \mathbf{H}' = \frac{4 \pi}{c} \sigma \mathbf{E}' \]

\[ \nabla \times (\frac{c}{4 \pi \sigma} \nabla \times \mathbf{H}) = -\frac{1}{c} \frac{\partial \mathbf{H}'}{\partial t} \]

\[ \nabla \times (\mathbf{G} \times \mathbf{H}) = -\frac{4 \pi}{c^2} \frac{\partial \mathbf{H}}{\partial t} \]

\[ \nabla^2 \mathbf{H} = \frac{4 \pi}{c^2} \frac{\partial \mathbf{H}}{\partial t} \]

\[ \nabla^2 \mathbf{H} = \frac{4 \pi \sigma}{c^2} \frac{\partial \mathbf{H}}{\partial t} = 0 \]

(64.17) same as eqn. for \( \mathbf{H} \) in conductor at rest
outside the body

\[ \nabla \times H = \frac{1}{c} \frac{dE}{dt} \]
\[ \nabla \times E = -\frac{1}{c} \frac{dH}{dt} \quad \text{(max's eqns in vacuum)} \]

\[ \nabla \times (\nabla \times H) = \nabla (\nabla \cdot H) - \nabla^2 H = -\frac{1}{c^2} \frac{d^2}{dt^2} H \]

\[ \nabla \cdot (\nabla \times H) = 0 \]

\[ \frac{\omega^2}{c^2} \approx \frac{1}{\lambda^2} \quad \text{small} \]

\[ \therefore \quad \nabla^2 H = 0 \quad \text{outside the body} \]

\[ \text{(but smaller than \lambda away)} \]

\[ \nabla^2 H' = \nabla^2 H - \frac{2mc}{e} \nabla^2 \rho = \nabla^2 H = 0 \]

on the surface of the conductor \( H' \), like \( H \) is continuous. (see (58.8)). At "infinity"

\[ H \text{ tends to zero, so } H' \text{ tends to } -\frac{2mc}{e} \rho = \mathbf{H} \]

\( H' \text{ is found by subtracting } \mathbf{H} \text{ from } H' \)

\[ H' - \mathbf{H} = H + \mathbf{H} - \mathbf{H} = \mathbf{H} \]

variable \( H' \) produces electric currents (64.4 \( \rightarrow \) 64.2) in the conductor.

The magnetic moment in a simply-connected body in a NON uniformly rotating body, the effect appears as an e.m.f.,-The Stewart-Tolman effect.
Problem 3 Section 6.4 LL®L ECM.

We are asked to determine the current in a superconducting ring which ceases to rotate uniformly. The solution makes use of Eq. (54.5) for the magnetic flux through a superconducting ring with varying current and external field.

\[ \Phi_e + L \frac{dc}{dt} = \Phi_0, \text{ a constant.} \]

In order to arrive at this formula, briefly consider the contents of §§53, 54 from chapter VI on superconductivity.

§53 Magnetic properties of superconductors

At temperatures below the superconductivity transition point, many metals exhibit a complete lack of electric resistance to a steady current. This change in the metal's electrical properties is a consequence of its magnetic properties.

The magnetic field does not penetrate into a superconductor.

\[ B = 0 \text{ Throughout.} \]

\[ B = 0 \] does not hold in a thin (\( \approx 10^{-5} \text{ cm} \)) layer near the surface, nor in thin films of metal or small metallic particles.

The normal component of induction must be must be continuous at the boundary between two media (since \( D \cdot B = 0 \)). Thus the normal component of the external field must be zero, and the field outside a superconductor must be tangential to its surface.
In order to calculate the forces on a superconductor in a magnetic field, we calculate \( \sigma_{ik} \cdot n_z \), the force per unit area.

The Maxwell stress tensor:

\[
\sigma_{ik} = \frac{(H_i H_k - \frac{1}{2} H^2 \delta_{ik})}{4\pi}
\]

for a magnetic field in a vacuum.

\[
(F_s)_i = \sigma_{ik} n_k = \frac{1}{4\pi} \left( H_i H_k n_k - \frac{1}{2} H^2 \delta_{ik} n_k \right)
\]

\[
H \cdot n = 0
\]

\[
= -\frac{1}{8\pi} H^2 n_i
\]

\[
F_s = -\frac{1}{8\pi} H_e n
\]

Since \( n \) points outward from the surface, this force is compressive.

\[
\nabla \times b = \frac{1}{c} \frac{\partial E}{\partial t} + \frac{4\pi}{c} \rho v
\]

\[
\bar{b} = B \quad \frac{1}{c} \frac{\partial \bar{E}}{\partial t} = 0 \text{ since the mean field } \bar{E} = \bar{\bar{E}} \text{ is const.}
\]

\[
\nabla \times B = \frac{4\pi}{c} \rho v
\]

(a field in a vacuum abhors a superconducting medium)

Since \( B = 0 \) in the body of a superconductor, \( \nabla \times B = 0 \Rightarrow \frac{4\pi}{c} \rho v = 0 \) \( \Rightarrow \) no macroscopic current can flow inside a superconductor.

In an ordinary conductor, we separate the conduction current from \( \rho v \) and define the magnetization through

\[
\nabla \times M = \frac{\rho v}{c}
\]

(\( \nabla \times M \) conduction current removed)

see (29.6)
No such separation can be made for a superconductor, so neither $M$ nor $H = B - 4\pi M$ have physical significance here. Any electric current which flows in a supercond. must be a surface current.

\[ g = \oint \vec{P} \times d\vec{l} \quad (29.15) \]

may tend to some finite limit. From vacuum to superconductor

The charge passing per unit time per unit length across a line in the surface.

Since $\frac{4\pi}{c} \vec{P} = \nabla \times \vec{B}$, we have

\[ g = \frac{c}{4\pi} \oint d\vec{l} \nabla \times \vec{B} \]

\[ g_y = \frac{c}{4\pi} \int d\vec{x} \left( \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) \]

\[ = -\frac{c}{4\pi} \frac{d}{d\ell} \left( B_z \right)_2 - \left( B_z \right)_1 \]

\[ = -\frac{c}{4\pi} \frac{d}{d\ell} (B_z)_2 - (B_z)_1 \]

\[ g_y = -\frac{c}{4\pi} (B_z)_2 + \frac{c}{4\pi} (B_z)_1 \]

\[ (H_z)_2 \quad 0 \text{ inside the superconductor} \]
More generally,

\[ \mathbf{g} = \frac{\epsilon}{4\pi} \mathbf{n} \times \mathbf{H} - \mathbf{e} \quad (53.4) \]

A surface current can occur in any magnetized body, given by

\[ \mathbf{g} = \frac{\epsilon}{4\pi} \mathbf{n} \times (\mathbf{H}_e - \mathbf{B}). \]

The tangential component of \( \mathbf{H} \) is continuous in a normal conductor (since \( \mathbf{D} \times \mathbf{H} = \frac{4\pi}{c^2} \mathbf{J} \), where \( \mathbf{J} \) is the conduction current density).

\[ \mathbf{H} = \mathbf{B}/\mu, \text{ so } \mathbf{n} \times \mathbf{H}_e = \mathbf{n} \times \mathbf{H} = \mathbf{n} \times \mathbf{B}/\mu, \text{ hence } \]

\[ \mathbf{g} = \frac{\epsilon}{4\pi} \mathbf{n} \times (\mathbf{B}/\mu - \mathbf{B}) \]

\[ (53.5) \]

\[ \mathbf{g} = \frac{\epsilon}{4\pi} \mathbf{n} \times \mathbf{B} \left( \frac{1-\mu}{\mu} \right) \]

A fundamental difference appears when we consider the total current through a cross section of the body. In a normal conductor, the surface currents always balance so that the total current is zero. This is seen in (53.5) which relates the surface current density \( \mathbf{g} \) at a given point to the magnetic induction, and hence to \( \mathbf{g} \) at every other point on the surface. In superconductors, no relation (53.5)
has no meaning. For the transition from the ordinary state (with magnetic permeability $\mu$) to the superconducting state corresponds formally to $B \to 0 \; \mu \to 0$, so that the RHS of (53.5) and there is no condition which restricts the possible values of the current.

From (53.4), the currents flowing on the surface of a superconductor may amount to a nonzero total current.

A steady flow of current on a superconductor is possible even if no E-field is present. This means that no dissipation of energy occurs whose replacement would involve work being done by an external field. There is no resistance which is thus a necessary consequence of the superconductor's magnetic properties.
We continue our discussion by working through § 54 on the superconductivity current in an effort to derive Eq. (54.5), which we need for prob 3 of § 64.

Some properties of superconductors depend on their shape.

Simply connected? Then no steady current in the absence of an external magnetic field.

(we don't mean of field that is external to the body, but rather an applied field)

Surface current would produce a static field in a vacuum that vanishes at \( \infty \). Since \( \mathbf{D} \times \mathbf{H}_e = \mathbf{0} \), we can write \( \mathbf{H}_e = -\mathbf{D} \mathbf{g} \). By continuity of \( \mathbf{H} \mathbf{n} \), we'd have \( 0 = (\mathbf{H}_e) \mathbf{n} = (\mathbf{H}_e) \mathbf{n} = -\mathbf{D} \mathbf{g} \mathbf{n} \).

Thus \( \mathbf{H}_e = -\mathbf{D} \mathbf{g} \mathbf{n} \) on the surface and at infinity.

Consider the flux of \( \mathbf{H}_e \) through the surfaces enclosing any volume external to the (simply connected) superconductor.

\[
- \int_{\mathcal{S}_1} \mathbf{da} \cdot \mathbf{H}_e + \int_{\mathcal{S}_2} \mathbf{da} \cdot \mathbf{H}_e = \int_{\mathcal{V}} \mathbf{d}^{3}r \cdot \mathbf{D} \cdot \mathbf{H}_e = 0
\]

\( \mathbf{da} H_n = -\mathbf{da} \frac{\partial \mathbf{H}_e}{\partial n} = 0 \)

Since the surface \( \mathcal{S}_2 \) is arbitrary, \( \mathbf{H}_e = \mathbf{0} \) everywhere \(- \nabla \mathbf{g} = \mathbf{0} \) outside the superconductor, so \( \mathbf{g} = \text{constant} \).
An external magnetic field causes a current to flow on the surface of a simply-connected superconductor. These currents can give the body a nonzero total magnetic moment, which we use to define an auxiliary “magnetization” through

\[ \mathbf{M} = M \mathbf{V} \]

and a “magnetic field” \( \mathbf{H} = \mathbf{B} - 4\pi \mathbf{M} \).

Though these quantities lack their usual local significance.

For a superconducting ellipsoid

\[
(1-n) \mathbf{H} + n \mathbf{B} = \frac{n}{4}
\]

(see 29.14)

but \( B = 0 \), so

\[
\mathbf{H} = \frac{n}{4(1-n)}
\]  \hspace{1cm} (54.1)

\[
\mathbf{M} = \mathbf{V} \mathbf{M} = - \frac{\mathbf{V} \mathbf{H}}{4\pi} = - \frac{n}{4\pi(1-n)}
\]  \hspace{1cm} (54.2)

For a long cylinder in a magnetic field

\[
\mathbf{H} \times \mathbf{D} = 0 \]  \hspace{1cm} \text{(circuit shrunk sufficiently)}
\]

\[
\mathbf{H} = \mathbf{h}
\]

\[
\Rightarrow \mathbf{M} = 0
\]
\[ H = \frac{h}{r} \quad (\text{for a long cylinder}) \Rightarrow \frac{M}{h} = -\frac{1}{4\pi} \]

Saneo having a diamagnetic susceptibility \( \chi = -\frac{1}{4\pi} \)

He outside the ellipsoid is tangential to it.

Within the ellipsoid \( H = \frac{n}{r(1-n)} \), the tangential component must be continuous, so

\[ H_e = (H)_{\text{tangential}} = \frac{1}{1-n} (\frac{h}{r})_t = \frac{1}{1-n} h \sin\theta \quad (521,3) \]

On the equator,

\[ H_e = \frac{h}{1-n} \]

Since the currents which cause the "magnetiz'n" are the same as those which produce the total current, the momentum density of the "magnetizing" currents differs from the current density by a factor \( k \).

136.3) \( M_i = e \sigma \text{me} \ g(i) \text{e} \ L_{gyr}, i \Rightarrow \ g(i) = \frac{3h}{k} \text{for a superconductor} \)
Multiply-connected superconductors

- STEADY DISTRIBUTION OF SURFACE CURRENTS is possible even in the absence of an external field.
- Surface currents need not balance out; may result in a steady total superconductivity current, even in the absence of an external field.

Doubly connected body: a ring - state is determined if TOTAL CURRENT \( J \) is given.

Potential \( \phi \) determining \( H^{(e)} \) is now many-valued, changes by \( 4\pi I/e \) when we go around any closed path interlinking the ring.

\[
P \times H = \frac{4\pi}{c} \cdot \mathbf{d}F
\]

\[
\frac{4\pi}{c} \int \mathbf{d}a \cdot \mathbf{d}F = \int \mathbf{d}a \cdot \mathbf{d} \times H
\]

From the field near the surface of the ring, we can uniquely determine

The surface current distribution

- self-inductance completely determinable to get \( \mathbf{E} \)
- field current distribution

\( \mathbf{H}^{(c)} \) is the only field to current, hence field energy is all Peierls is
\[ \Phi = \Phi_0 - \frac{1}{2} \sum a_i \Phi_a \quad (33.20) \]

\[ \sum c = \frac{\Phi}{\Phi_0}, \quad \Phi_a/c = -\Phi_0/2c \]

\[ (33.8): \quad \Phi = \frac{1}{2c^2} \sum a_i \Phi_a^2 + \frac{1}{c^2} \sum a_i a_j \frac{\partial \Phi}{\partial a_j} \]

\[ \frac{\partial \Phi}{\partial a_i} = \frac{\partial \Phi}{\partial a_i} - \frac{\Phi_a}{c} \]

minimize \[ \Phi \rightarrow 0 = \frac{1}{c^2} \sum a_i \Phi_a + \frac{1}{c^2} \sum a_i a_j \frac{\partial \Phi}{\partial a_j} - \frac{\Phi_a}{c} \]

(33.22): \[ \Phi_a = \frac{1}{c} \sum a_i a_j \frac{\partial \Phi}{\partial a_j} \]

For a single circuit

\[ \Phi = \frac{1}{c} \sum a_i a_j \frac{\partial \Phi}{\partial a_j} \]

For a superconductor, the magnetic flux is meaningful for any thickness of the ring, not necessarily small. Since the field is tangential, the magnetic flux through any part of the surface of the ring is zero. The flux through any surface spanning the ring is the same.

The formula

\[ (34.4) \quad \Phi = \frac{\mathcal{L}}{c} \text{ remains valid} \]

defined in terms of the total energy of the magnetic field of the circuit \[ \int dV \frac{H^2}{2} \text{ outside} \]
Spanning the ring by a surface \( \mathcal{C} \)

\[
\int \mathbf{H} \cdot d\mathbf{A} / 8\pi = -\int H \cdot d\phi / 8\pi
\]

\[
= \oint (\nabla \cdot \mathbf{H}) \phi \ d\mathbf{A} / 8\pi - \oint H_n \phi \ d\mathbf{A} / 8\pi
\]

This integral is over an infinitely remote surface, the surface of Roning, and the two sides of the surface \( \mathcal{C} \).

\[
= \oint \left( H_0 (\phi_2 - \phi_1) \right) \ d\mathbf{A} / 8\pi
\]

\[
\frac{4\pi J}{c}
\]

\[
\int \mathbf{H} \cdot d\mathbf{A} / 8\pi = \frac{H}{2c}
\]

\[
\frac{1}{2} c^2 L J^2 \quad \text{by definition of self-inductance}
\]

\[
\therefore \quad \frac{LJ}{c} = \Phi
\]

which is (64.4) and justifies the claim that it applies to SC circuit.