The **Weyl representation** expresses an operator $A(R,P)$ as a superposition of **displacement operators**

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$$A(R,P) = \int \frac{d\sigma\, d\tau}{(2\pi\hbar)^{3N}}\, e^{i[\sigma\cdot R + \tau\cdot P]/\hbar}\, \alpha(\sigma,\tau) \quad ;$$

I've defined this expansion in such a way that **α has the same units as A.**

To find the "expansion coefficients" α, we evaluate

$$\mathrm{Tr}\left[e^{-i(\bar\sigma\cdot R + \bar\tau\cdot P)/\hbar} A\right] = \int \frac{d\sigma\, d\tau}{(2\pi\hbar)^{3N}}\, \alpha(\sigma,\tau)$$

$$\times\, \underbrace{\mathrm{Tr}\left[e^{i[\sigma\cdot R + \tau\cdot P]/\hbar}\, e^{-i[\bar\sigma\cdot R + \bar\tau\cdot P]/\hbar}\right]}$$

$$\int dr\, e^{i(\sigma-\bar\sigma)\cdot r/\hbar}\, \underbrace{\langle r|\, e^{i\tau\cdot P/\hbar}\, e^{-i\bar\tau\cdot P/\hbar}\, |r\rangle}_{\langle r+\tau-\bar\tau|r\rangle \,=\, \delta(\tau-\bar\tau)}\, e^{\frac{i}{2\hbar}(\sigma\cdot\tau - \bar\sigma\cdot\bar\tau)}$$

$$= (2\pi\hbar)^{3N}\, \delta(\sigma-\bar\sigma)\, \delta(\tau-\bar\tau)$$

$$= \alpha(\bar\sigma,\bar\tau)$$

$$\mathrm{Tr}\left[e^{i(\sigma\cdot R + \tau\cdot P)/\hbar}\, e^{-i(\bar\sigma\cdot R + \bar\tau\cdot P)/\hbar}\right] = (2\pi\hbar)^{3N}\, \delta(\sigma-\bar\sigma)\, \delta(\tau-\bar\tau)$$

$$\alpha(\sigma,\tau) = \mathrm{Tr}\left[e^{-i(\sigma\cdot R + \tau\cdot P)/\hbar} A(R,P)\right]$$

Position matrix elements

$$\langle r|A|\bar{r}\rangle = \int \frac{d\sigma\, d\tau}{(2\pi\hbar)^{3N}}\, e^{i\sigma\cdot r/\hbar} \underbrace{\langle r+\tau|\bar{r}\rangle}_{\delta(\tau-\bar{r}+r)} e^{i\sigma\cdot\tau/2\hbar}\, \alpha(\sigma,\tau)$$

$$= \frac{1}{(2\pi\hbar)^{3N}} \int d\sigma\, e^{i\sigma\cdot r/\hbar}\, e^{i\sigma\cdot(\bar{r}-r)/2\hbar}\, \alpha(\sigma,\bar{r}-r)$$

$$\boxed{\langle r|A|\bar{r}\rangle = \frac{1}{(2\pi\hbar)^{3N}} \int d\sigma\, e^{i\sigma\cdot(r+\bar{r})/2\hbar}\, \alpha(\sigma,\bar{r}-r)}$$

The Wigner equivalent of $A(R,P)$ is defined as

$$A_w(r,p) = \int d\tau\, e^{ip\cdot\tau/\hbar} \langle r-\tau/2|A(R,P)|r+\tau/2\rangle$$

$$= \int d\tau\, e^{ip\cdot\tau/\hbar} \frac{1}{(2\pi\hbar)^{3N}} \int d\sigma\, e^{i\sigma\cdot r/\hbar}\, \alpha(\sigma,\tau)$$

$$\boxed{A_w(r,p) = \int \frac{d\sigma\, d\tau}{(2\pi\hbar)^{3N}}\, e^{i(\sigma\cdot r+\tau\cdot p)/\hbar}\, \alpha(\sigma,\tau)}$$

Thus, **we obtain $A_w(r,p)$ by replacing (R,P) with (r,p) in the Weyl expansion of A.**

In order to obtain the Weyl components of a product, we must evaluate

$$\alpha\beta(\sigma,\tau) = \mathrm{Tr}\left[e^{-i[\sigma\cdot R+\tau\cdot P]/\hbar} AB\right]$$

$$= \int \frac{d\bar{\sigma}\, d\bar{\tau}}{(2\pi\hbar)^{3N}} \int \frac{d\bar{\bar{\sigma}}\, d\bar{\bar{\tau}}}{(2\pi\hbar)^{3N}}\, \alpha(\bar{\sigma},\bar{\tau})\, \beta(\bar{\bar{\sigma}},\bar{\bar{\tau}})$$

$$\times \mathrm{Tr}\left[e^{-i(\sigma\cdot R+\tau\cdot P)/\hbar}\, e^{i(\bar{\sigma}\cdot R+\bar{\tau}\cdot P)/\hbar}\, e^{i(\bar{\bar{\sigma}}\cdot R+\bar{\bar{\tau}}\cdot P)/\hbar}\right]$$

$$e^{-i(\sigma\cdot R+\tau\cdot P)/\hbar}\, e^{i(\bar{\sigma}\cdot R+\bar{\tau}\cdot P)/\hbar}\, e^{i(\bar{\bar{\sigma}}\cdot R+\bar{\bar{\tau}}\cdot P)/\hbar}$$

$$= e^{-i\sigma\cdot R/\hbar}\, e^{-i\tau\cdot P/\hbar}\, e^{\frac{i}{2\hbar}\sigma\cdot\tau}\, e^{i\bar{\sigma}\cdot R/\hbar}\, e^{i\bar{\tau}\cdot P/\hbar}\, e^{\frac{i}{2\hbar}\bar{\sigma}\cdot\bar{\tau}}$$

$$\times\, e^{i\bar{\bar{\tau}}\cdot P/\hbar}\, e^{i\bar{\bar{\sigma}}\cdot R/\hbar}\, e^{-\frac{i}{2\hbar}\bar{\bar{\tau}}\cdot\bar{\bar{\sigma}}}$$

$$= e^{\frac{i}{2\hbar}(\sigma\cdot\tau+\bar{\sigma}\cdot\bar{\tau}-\bar{\bar{\tau}}\cdot\bar{\bar{\sigma}})}\, e^{-\frac{i}{\hbar}\sigma\cdot R}\, e^{\frac{i}{\hbar}(\bar{\tau}+\bar{\bar{\tau}}-\tau)\cdot P}$$

$$\times\, e^{\frac{i}{\hbar}\bar{\sigma}\cdot\underbrace{\left(e^{-\frac{i}{\hbar}(\bar{\tau}+\bar{\bar{\tau}})\cdot P}\, R\, e^{\frac{i}{\hbar}(\bar{\tau}+\bar{\bar{\tau}})\cdot P}\right)}_{R-\bar{\tau}-\bar{\bar{\tau}}}}\, e^{\frac{i}{\hbar}\bar{\bar{\sigma}}\cdot R}$$

$$= e^{\frac{i}{2\hbar}(\sigma\cdot\tau+\bar{\sigma}\cdot\bar{\tau}-\bar{\bar{\sigma}}\cdot\bar{\bar{\tau}})}\, e^{-\frac{i}{\hbar}\sigma\cdot R}\, e^{\frac{i}{\hbar}(\bar{\tau}+\bar{\bar{\tau}}-\tau)\cdot P}$$

$$\times\, e^{\frac{i}{\hbar}\bar{\sigma}\cdot(R-\bar{\tau}-\bar{\bar{\tau}})}\, e^{\frac{i}{\hbar}\bar{\bar{\sigma}}\cdot R}$$

$$= e^{\frac{i}{2\hbar}(\sigma\cdot\tau-\bar{\sigma}\cdot\bar{\tau}-\bar{\bar{\sigma}}\cdot\bar{\bar{\tau}})}\, e^{-\frac{i}{\hbar}\bar{\sigma}\cdot\bar{\bar{\tau}}}$$

$$\times\, e^{-\frac{i}{\hbar}\sigma\cdot R}\, e^{\frac{i}{\hbar}(\bar{\tau}+\bar{\bar{\tau}}-\tau)\cdot P}\, e^{\frac{i}{\hbar}(\bar{\sigma}+\bar{\bar{\sigma}})\cdot R}$$

Substituting on p. 2 leads to

$$\alpha\beta(\sigma,\tau) = \int \frac{d\bar\sigma\, d\bar\tau}{(2\pi\hbar)^{3N}} \int \frac{d\bar{\bar\sigma}\, d\bar{\bar\tau}}{(2\pi\hbar)^{3N}}\, \alpha(\bar\sigma,\bar\tau)\, \beta(\bar{\bar\sigma},\bar{\bar\tau})$$

$$\times\, e^{\frac{i}{2\hbar}(\sigma\cdot\tau - \bar\sigma\cdot\bar\tau - \bar{\bar\sigma}\cdot\bar{\bar\tau})}\; e^{-\frac{i}{\hbar}\bar\sigma\cdot\bar{\bar\tau}}$$

$$\times \int dr\, e^{\frac{i}{\hbar}(\bar\sigma + \bar{\bar\sigma} - \sigma)\cdot r}\quad \delta(\tau - \bar\tau - \bar{\bar\tau})$$

$$\boxed{\alpha\beta(\sigma,\tau) = \int \frac{d\bar\sigma\, d\bar\tau}{(2\pi\hbar)^{3N}}\, \alpha(\bar\sigma,\bar\tau)\, \beta(\sigma-\bar\sigma, \tau-\bar\tau)\, e^{\frac{i}{2\hbar}(\sigma\cdot\bar\tau - \bar\sigma\cdot\tau)}}$$

OK, agrees w/ 6 Apr 14

Imre et al. don't state This pretty result. Does someone else???

For the Wigner equivalent of AB, we get

$$(AB)_W(r,p) = \int \frac{d\bar\sigma\, d\bar\tau}{(2\pi\hbar)^{3N}} \int \frac{d\sigma\, d\tau}{(2\pi\hbar)^{3N}}\, e^{i[\sigma\cdot r + \tau\cdot p]/\hbar}\; e^{\frac{i}{2\hbar}(\sigma\cdot\bar\tau - \bar\sigma\cdot\tau)}$$

$$\times\, \alpha(\bar\sigma,\bar\tau)\, \beta(\sigma-\bar\sigma, \tau-\bar\tau)\,.$$

Let $\bar{\bar\sigma} = \sigma - \bar\sigma$ & $\bar{\bar\tau} = \tau - \bar\tau$ $(\Longrightarrow \sigma = \bar\sigma + \bar{\bar\sigma}$ & $\tau = \bar\tau + \bar{\bar\tau})$

$d\bar{\bar\sigma} = d\sigma$ & $d\bar{\bar\tau} = d\tau$.

$$(AB)_W(r,p) = \int \frac{d\bar{\sigma}\, d\bar{\tau}}{(2\pi\hbar)^{3N}} \int \frac{d\bar{\bar{\sigma}}\, d\bar{\bar{\tau}}}{(2\pi\hbar)^{3N}}\, e^{\frac{i}{\hbar}[\bar{\sigma}\cdot r + \bar{\tau}\cdot p]}\, \alpha(\bar{\sigma},\bar{\tau})$$

$$\times\, e^{\frac{i}{\hbar}[\bar{\bar{\sigma}}\cdot r + \bar{\bar{\tau}}\cdot p]}\, \beta(\bar{\bar{\sigma}},\bar{\bar{\tau}})\, e^{\frac{i}{2\hbar}[\bar{\tau}\cdot\bar{\bar{\sigma}} - \bar{\sigma}\cdot\bar{\bar{\tau}}]}$$

$$\boxed{(AB)_W(r,p) = A_W(r,p)\, e^{\frac{\hbar\Lambda}{2i}}\, B_W(r,p)},$$

where $\boxed{\Lambda = \overleftarrow{\nabla}_p \cdot \overrightarrow{\nabla}_r - \overleftarrow{\nabla}_r \cdot \overrightarrow{\nabla}_p}$ is the **Poisson-bracket operator.**

According to İmre and co-workers (1967), this expression is due to Groenwold (1946). It's used by Kapral & Ciccotti (1999) to obtain an $\hbar$-expansion for the partial Wigner transform of a product, and their analysis is in turn the basis for Petit and Subotnik's 2014 surface-hopping treatment of linear absorption.