Multimode laser model with coupled cavities and quantum noise

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A stochastic, semiclassical model is developed for a multimode, homogeneously broadened laser with rapid dipole dephasing, appropriate for semiconductor, Ti:sapphire, or dye lasers. The theory self-consistently incorporates population dynamics including temporal beating effects and relaxation oscillations, spatial hole burning, coherent-wave mixing, and quantum noise. The model is valid for single- and compound-cavity lasers in which the mode frequencies are well defined. We pay particular attention to finding a useful mode basis in the case that the gain medium does not completely fill the cavity. This situation can lead to coupled-cavity effects. For typical systems the model is valid for pump rates up to several times threshold and is tractable for numerical simulations. The theoretical development described in this paper is applied to an experimental system in a companion paper (J. Opt. Soc. Am. B **14**, 180 (1997)]. © 1997 Optical Society of America. [S0740-3224(97)01401-X]

1. INTRODUCTION

The study of laser instabilities, including the turn-on, as well as other dynamics is rich and varied and as old as the laser. Fully quantum-mechanical theories of optical amplifiers usually reduce in practice to those derived semiclassically.[1–4] Thus the first semiclassical theories[5,6] are still commonly employed today with good success, often with the quantum-mechanical spontaneous-emission noise represented as a random initial condition.[7–12] More-detailed models of dynamics are obtained when the spontaneous emission is treated as a dynamic, random variable (Ref. 9, Chap. 20; Refs. 13–15). Single-mode laser theories, including dynamic noise terms, are mature and have been applied extensively.[3,16,17] However, multimode theories, which are necessarily more complicated than single-mode theories, are more varied and specialized.[17] For instance, in semiconductor laser treatments the spatial variation of the population inversion is often neglected so that spatial hole-burning and coherent-wave mixing effects are lost, but the correct noise is used (see, e.g., Refs. 13 and 15). This approximation, though it is usually adequate because of the fast spatial diffusion of carriers in the semiconductor laser, has measurable limits.[18–21] The most obvious is that such models are not truly multimode: Only a single mode survives in continuous-wave (cw) operation. If the spatial dependence of the inversion is not neglected, then multimode cw operation can be modeled (see, e.g., Refs. 14 and 22). These treatments typically neglect the dynamical aspect of the noise[10,11,23] or borrow from single-mode theories.[14,24]

The motivation for the study described here[25] is the desire to model the dynamics of individual modes in a multimode laser, including turn-on-time statistics[26] and mode hopping.[27] We have applied the theory developed here for these applications in the case of a short-cavity dye laser with good success, improving the simulation results presented in Ref. 26. A companion paper to this one[28] describes these newer experimental and simulation results.

In this paper we develop a homogeneously broadened multimode laser model from first principles that includes quantum-noise terms that are consistent with the assumptions used in deriving the deterministic aspects of the theory. The dynamic variables are the electric field inside the cavity, the projections of the population inversion onto the products of spatial field modes, and the polarization (dipole moment density) of the gain medium. The gain medium is assumed to be a system that can be approximated by two energy levels.[29] The quantum noises are modeled with stochastic Langevin terms in the equations of motion, which are consistent with a quantum-mechanical treatment under typical conditions (as defined in Ref. 4). When the polarization is adiabatically eliminated the resulting equations of motion for the field and inversion can exhibit relaxation oscillations and other well-known laser traits. Terms that represent coherent, nondegenerate four-wave mixing (NDFWM), spatial hole burning, and frequency-dependent saturation arise in a self-consistent fashion. If the inversion is adiabatically eliminated, the well-known third-order field equations (e.g., in Ref. 9, Chap. 9; Ref. 14) result. The spatial dependence of the inversion and the pump is accounted for, and no functional form is assumed for them, *a priori*. The model is valid for compound cavities that are coupled strongly enough that frequency pushing effects are small (see Ref. 30 for a classification of coupled-cavity lasers based on the strength of the coupling) as well as for simpler configurations in which the gain medium fills the cavity.

The stochastic noises driving the individual field modes are shown to be partially correlated. The uncorrelated part of the noise depends on the spatial average and the standing spatial wave (spatial hole-burning) parts of the inversion. The correlated noise strength is proportional to the amplitude of the running spatial wave in the inversion that is due to NDFWM and has not been analyzed before.

The model presented here is not a comprehensive one in that there are many well-known phenomena whose effects are neglected. Among them are (a) gain-medium specific properties such as triplet quenching in dyes[31–34] and spatial diffusion and linewidth enhancement in semiconductors,[13,21] (b) time-varying constants (e.g., conductivity and dielectric constants) (see Refs. 35 and 36), (c) limitations of the laser equations because of atomic-cavity detuning (see, e.g., Refs. 37–40), (d) large changes in the refractive index that are due to intracavity intensity (see, e.g., Ref. 9, Chap. 8; Ref. 19) or carrier concentration fluctuations,[20,41] (e) noise that is not Markovian with a correlation function that does not decay exponentially[42] (an overview of the theory of laser noise is given by Lax in Ref. 3), (f) multiplicative noise that is due to pump fluctuations,[43,44] and (g) spectral hole-burning effects including NDFWM contributions.[45,46] Despite these shortcomings, the model presented yields good agreement with the short-cavity dye-laser measurements presented in the companion paper[28] and in Refs. 25–27, and this comparison is the primary motivation for what follows.

2. MATTER–FIELD EQUATIONS

In the two-level approximation of a four-level lasing medium the radiation and molecular dynamics can be described in time and space by stochastic, semiclassical Maxwell–Bloch equations whose noise terms are fully compatible with quantum mechanics. In the rotating-wave and the slowly varying envelope approximations these coupled partial differential equations are given in Ref. 4 in Gaussian units for the two-level model. If the equations are slightly modified to model four-level atoms[32] and local field corrections are considered,[47,48] the result is

$$\frac{\partial}{\partial t} P(\mathbf{r}, t) = -(\gamma_\perp + i\delta)P(\mathbf{r}, t) - \frac{i\mu^2}{\hbar} W(\mathbf{r}, t)A(\mathbf{r}, t) + \mu\rho\eta(\mathbf{r}, t), \tag{2.1}$$

$$\frac{\partial}{\partial t} W(\mathbf{r}, t) = \gamma_\parallel[R(\mathbf{r}, t) - W(\mathbf{r}, t)] + \frac{i}{4\hbar}[A(\mathbf{r}, t)P^*(\mathbf{r}, t) - A^*(\mathbf{r}, t)P(\mathbf{r}, t)] + \rho\theta(\mathbf{r}, t), \tag{2.2}$$

$$\left[c^2\nabla^2 - \epsilon(\mathbf{r})\frac{\partial^2}{\partial t^2} - \kappa(\mathbf{r})\frac{\partial}{\partial t}\right]E(\mathbf{r}, t)\exp(-i\omega t) = 4\pi\left[\frac{\epsilon(\mathbf{r}) + 2}{3}\right]\frac{\partial^2}{\partial t^2}P(\mathbf{r}, t)\exp(-i\omega t). \tag{2.3}$$

The total electric field in the slowly varying envelope approximation is given by $[E(\mathbf{r}, t)\exp(-i\omega t) + \text{c.c.}]/2$. The slowly varying part of the polarization, $P(\mathbf{r}, t)$ (dipole moment per unit volume), is due to the real transition dipole moment μ, decays at the collisional dephasing rate $\gamma_\perp$, and drives the macroscopic field $E(\mathbf{r}, t)$. The field local to the atoms, $A(\mathbf{r}, t)$, is different from the macroscopic field because of the polarization of the host medium, which has a dielectric constant $\epsilon(\mathbf{r})$. Variations in index of refraction $n(\mathbf{r})$ are represented by the dielectric constant, $\epsilon(\mathbf{r}) = n^2(\mathbf{r})$. The field is dissipated by the distributed loss $\kappa(\mathbf{r})$ that models both intracavity absorption and the output coupling losses that are particular to a cavity geometry (these generally depend on frequency). The center field frequency ω and the peak of the fluorescent emission profile ω_{21} differ by a detuning $\delta = \omega_{21} - \omega$. The inversion $W(\mathbf{r}, t)$, spontaneously decaying at the rate $\gamma_\parallel$, is replenished at a pump rate $R(\mathbf{r}, t)$, which maintains the system above threshold. The number density of lasing centers in the pumped region is ρ and is assumed uniform. The terms $\eta(\mathbf{r}, t)$ and $\theta(\mathbf{r}, t)$ are Langevin forces that represent the intrinsic noise that is due to quantum fluctuations of the polarization and the inversion, respectively. They are assumed to be independent complex Gaussian random processes with zero mean, are δ correlated in time and space, and are treated stochastically in the Ito sense.[4,49] We neglect the intrinsic field noise because it is due to thermal light that is insignificant in systems operating at optical frequencies. The correlation functions for the polarization and inversion noises are obtained from quantum reservoir theory and can be written (neglecting the nonclassical and pump shot-noise terms given in Ref. 4) as

$$\langle \eta^*(\mathbf{r}', t')\eta(\mathbf{r}, t)\rangle = 8\gamma_\perp W(\mathbf{r}, t)\delta(t' - t)\delta^3(\mathbf{r}' - \mathbf{r})/\rho^2, \tag{2.4}$$

$$\langle \theta(\mathbf{r}', t')\theta(\mathbf{r}, t)\rangle = 2\gamma_\parallel\delta(t' - t)\delta^3(\mathbf{r}' - \mathbf{r})/\rho, \tag{2.5}$$

with $\langle \eta(\mathbf{r}', t')\eta(\mathbf{r}, t)\rangle = \langle \eta(\mathbf{r}, t)\rangle = \langle \theta(\mathbf{r}, t)\rangle = 0$.

Solving Eqs. (2.1)–(2.3) requires a connection between the local field $A(\mathbf{r}, t)$ and the macroscopic field $E(\mathbf{r}, t)$. Their relation, to good approximation, is given by the Lorenz–Lorentz approximation[47,48]

$$A(\mathbf{r}, t) \approx \left[\frac{\epsilon(\mathbf{r}) + 2}{3}\right]E(\mathbf{r}, t). \tag{2.6}$$

If relation (2.6) is inserted into Eqs. (2.1)–(2.3) with the redefinitions $\{[\epsilon(\mathbf{r}) + 2]/3\}P(\mathbf{r}, t) \mapsto P(\mathbf{r}, t)$ and $\{[\epsilon(\mathbf{r}) + 2]/3\}\mu \mapsto \mu(\mathbf{r})$ the resulting Maxwell–Bloch equations (with linear local field corrections) that describe the relation of the atomic and optical dynamics read as

$$\frac{\partial}{\partial t} P(\mathbf{r}, t) = -(\gamma_\perp + i\delta)P(\mathbf{r}, t) - \frac{i\mu^2(\mathbf{r})}{\hbar} W(\mathbf{r}, t)E(\mathbf{r}, t) + \mu(\mathbf{r})\rho\eta(\mathbf{r}, t), \tag{2.7}$$

$$\frac{\partial}{\partial t} W(\mathbf{r}, t) = \gamma_\parallel[R(\mathbf{r}, t) - W(\mathbf{r}, t)] + \frac{i}{4\hbar}[E(\mathbf{r}, t)P^*(\mathbf{r}, t) - E^*(\mathbf{r}, t)P(\mathbf{r}, t)] + \rho\theta(\mathbf{r}, t), \tag{2.8}$$

$$\left[c^2\nabla^2 - \epsilon(\mathbf{r})\frac{\partial^2}{\partial t^2} - \kappa(\mathbf{r})\frac{\partial}{\partial t}\right]E(\mathbf{r}, t)\exp(-i\omega t) = 4\pi\frac{\partial^2}{\partial t^2}P(\mathbf{r}, t)\exp(-i\omega t). \quad (2.9)$$

The dynamical variables are now the macroscopic electric field $E(\mathbf{r}, t)$, the population inversion density $W(\mathbf{r}, t)$, and the polarization (i.e., the scaled dipole moment density) $P(\mathbf{r}, t)$. Equations (2.7)–(2.9) form a general description of the noise-influenced dynamics of coupled matter–field systems (e.g., optical amplifiers). The description is consistent with quantum mechanics, at least when nonclassical effects such as squeezing and sub-Poisson statistics are absent. They conserve energy (with pump and losses neglected), as do the equations that follow. We now specialize to the case of a resonant laser cavity.

3. LASER EQUATIONS

The laser system depicted in Fig. 1 is characterized by a cavity of optical length L_C and volume V_C and a gain medium with optical length L_G and volume V_G. Volumes correspond to the optically active regions and are defined by the particular geometry of the cavity.[32] The intracavity interface reflectivities are $r_1(\omega)$ and $r_2(\omega)$, and the output coupler reflectivities are $\Re_1(\omega)$ and $\Re_2(\omega)$, each of which is a function of frequency. The location of the gain medium within the cavity is arbitrary. When $\Re_1$ and $\Re_2$ are near unity the boundary conditions of Maxwell's equations suggest a natural Fourier expansion of the electric field into an orthonormal spatial basis $u_j(\mathbf{r})$:

$$E(\mathbf{r}, t) = \sum_k u_k(\mathbf{r})E_k(t)\exp(-i\Delta_k t), \quad (3.1)$$

where the expansion sum index range, for example, is $\pm(M-1)/2$, with M being the number of modes considered. The detuning of the jth-mode empty-cavity eigenfrequency ω_j from the center field-mode emission frequency is given by $\Delta_j = \omega_j - \omega$. (In the special case when the refractive index is uniform throughout the cavity, $\omega_j = jc\pi/L_C$.) The average separation of the mode frequencies (the average free spectral range) is $\bar{\Delta}$. The mode with $k = 0$ in Eq. (3.1) is the one whose frequency is closest to the atomic resonance frequency ω_{21}. Thus Δ_0 is the smallest of the detunings Δ_k and is less than $\bar{\Delta}/2$.

In general, the active gain medium will introduce shifts in the lasing frequencies from the empty-cavity frequencies. This effect turns out to be insignificantly small under the conditions considered in this study but is included in the dynamical evolution of the variables.

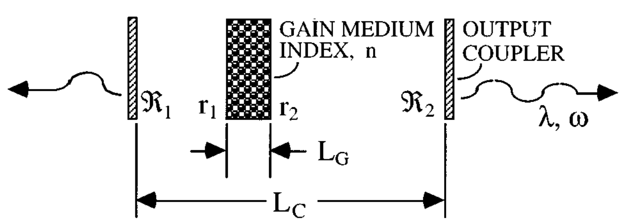

Fig. 1. Laser system.

The mean number of photons in the jth mode, $n_j(t)$, is related to the field-mode amplitudes by $n_j(t) = |E_j(t)|^2/8\pi\hbar\omega_j$ (assuming negligible frequency pulling or pushing).

One finds particular mode functions and allowed frequencies by applying the boundary conditions given by Maxwell's equations to the interfaces depicted in Fig. 1. The modes satisfy the Helmholtz equation

$$[c^2\nabla^2 + \epsilon(\mathbf{r})\omega_j^2]u_j(\mathbf{r}) = 0 \quad (3.2)$$

and are orthonormal over the cavity when the dielectric constant is included in the integral[50–54]:

$$\int_C \mathrm{d}^3 r\,\epsilon(\mathbf{r})u_j^*(\mathbf{r})u_k(\mathbf{r}) = \delta_{j,k}, \quad (3.3)$$

where the subscript C on the spatial integral indicates that the integration is performed over the entire cavity. It is important to note that the mode functions need not form a mathematically complete basis. Maxwell's equations (and the associated boundary conditions), depending on the choice of coordinates, reduce the number of modes necessary to describe the electric field completely.

Thus the field-mode amplitudes of Eq. (3.1) are given by

$$E_k(t) = \exp(i\Delta_k t)\int_C \mathrm{d}^3 r\,\epsilon(\mathbf{r})u_k^*(\mathbf{r})E(\mathbf{r}, t). \quad (3.4)$$

We assume that $E_k(t)$ is slow with respect to the longitudinal mode beat terms $\exp(-i\bar{\Delta}t)$. Note that this assumption may break down when the laser system is operated many times above threshold.[55–57]

If the temporal field coefficients $E_k(t)$ are slowly varying, then it is reasonable to expand the polarization into a spatial Fourier series that uses the same basis as the field,

$$P(\mathbf{r}, t) = \sum_k u_k(\mathbf{r})P_k(t)\exp(-i\Delta_k t), \quad (3.5)$$

where $P_k(t)$ is defined similarly to $E_k(t)$ in Eq. (3.4). Inspection of Eq. (3.5), keeping in mind that the polarization is zero everywhere outside the gain medium whereas $u_j(\mathbf{r})$ is not, shows that the $P_k(t)$ terms cannot be slowly varying if the expansion defines the gain medium spatially at all times, as it must. We thus expand $P_k(t)$ in a temporal Fourier series as

$$P_k(t) = \sum_q P_k^q(t)\exp[-i(\Delta_q - \Delta_k)t], \quad (3.6)$$

with the result that the coefficients $P_k^q(t)$ are slow (as long as $\delta \ll \bar{\Delta}$).

Thus, although a single expansion adequately describes the electric field with slowly varying coefficients, a double expansion of the polarization into a spatial and temporal series ensures that (1) the geometric extent of the gain medium can be well defined by the spatial series even when relatively few modes (e.g., single mode) are necessary to define the electric field and (2) the polarization coefficients $P_k^q(t)$ are slowly varying, unlike $P_k(t)$ in the single expansion of Eq. (3.5). This new expansion scheme is key to the following development.

Inasmuch as there is no general relationship between the inversion and the field, the inversion cannot be expanded as the polarization was, without loss of generality, unless the basis $u_j(\mathbf{r})$ is chosen so that it forms a rigorously complete basis. However, doing this would require that a specific functional form of the spatial dependence of the inversion be chosen, thus limiting the applicability of the laser model, and is usually a difficult task in practice.[17] Therefore, in this analysis, we expand the inversion only in a temporal series, thus avoiding having to choose a spatial dependence for the inversion. We find that this choice does not limit the laser model. Thus we write

$$W(\mathbf{r},\, t) = \sum_p W^p(\mathbf{r},\, t)\exp(-i\Delta_p t), \tag{3.7}$$

where, as for the polarization expansion, the coefficients $W^{p\neq 0}(\mathbf{r},\, t)$ cannot be slowly varying in time because the inversion is zero everywhere outside the gain medium. This allows for population oscillations at the intermodal beat frequencies (Chap. 9 of Ref. 9).

We simplify further analysis by assuming that fast terms [i.e., those with variations faster than $\exp(i\bar{\Delta}t)$] are small and so can be neglected, that the variations in the mode frequency separation are small compared with the mean separation, and that $\epsilon(\mathbf{r})$ is uniform within the gain medium: $\epsilon(\mathbf{r}) \mapsto \epsilon$, a constant within the gain-medium volume V_G and unity elsewhere.

The first assumption is reasonable in systems in which the pump rate is low so that the bandwidth of the dynamics is naturally limited by critical slowing[43,58] and when the cavity is short enough that the frequencies of the mode beats are much larger than the field or inversion decay rates. We call this a short-cavity approximation (SCA) because the free spectral range scales with the inverse of the cavity length. We use the SCA to separate terms in the equations of motion [Eqs. (2.7)–(2.9)] that result from using the expansions of Eqs. (3.1) and (3.5)–(3.7) by integrating the equations of motion over time steps $\tau \gg 1/\bar{\Delta}$. The second assumption allows us to treat residual mode beating (e.g., beats between beats) as slow. The third assumption simplifies the ensuing mathematics considerably and applies reasonably well to many systems.

We can find the equation of motion for the polarization components $P_j^q(t)$ by expanding Eq. (2.7), using Eqs. (3.1), (3.5), and (3.6), and applying the integral $\epsilon \int_G \mathrm{d}^3 r u_j^*(\mathbf{r})$ and the SCA (the subscript G indicates an integral only over the gain medium), with the result that

$$\frac{\mathrm{d}}{\mathrm{d}t} P_j^q(t) = -\beta_q P_j^q(t) - \frac{i\mu^2}{\hbar} \sum_k f_{q,k}^- E_{q-k}(t) W_{j,q-k}^k(t) + Q_j^q(t), \tag{3.8}$$

where $\beta_q = \gamma_\perp + i(\delta - \Delta_q)$ represents the damping of the polarization and the detuning of the cavity and the gain-medium resonances. The residual intermodal beat oscillations are represented by

$$f_{q,k}^\pm(t) \equiv \exp[i(\Delta_q \pm \Delta_k - \Delta_{q\pm k})t], \tag{3.9}$$

which has a value of 1 when q or k is 0 or if the modes are uniformly spaced (i.e., $\Delta_q = q\Delta$). The beats are slowly varying if the longitudinal-mode separation (free spectral range) is large compared with the nonuniformity of the mode spacing. We limit our analysis to these cases that correspond either to a single-cavity laser (i.e., the cavity is filled by the gain medium, $L_G = L_C$) or to a laser that is strongly coupled to an external cavity via the small intracavity interface reflectivities r_1 and r_2 (Tkach and Chaplyry in Ref. 30 define and classify regimes of coupling in compound-cavity laser systems).

The polarization noise in Eq. (3.8) is given by a coarse-graining integral over the noise term $\eta(\mathbf{r}, t)$:

$$Q_j^q(t) = \frac{\mu\rho\epsilon}{\tau} \int_{t-\tau}^t \mathrm{d}t' \exp(i\Delta_q t') \textstyle\int_G \mathrm{d}^3 r u_j^*(\mathbf{r})\eta(\mathbf{r},\, t'), \tag{3.10}$$

where τ is a time interval greater than $1/\Delta$ (beat period) but less than other dynamical time scales.

The inversion is now represented in Eq. (3.8) by $W_{j,k}^p(t)$, the projection of the temporal expansion coefficients $W^p(\mathbf{r}, t)$ onto pairs of spatial field modes within the gain medium[59]:

$$W_{j,k}^p(t) \equiv \epsilon \int_G \mathrm{d}^3 r u_j^*(\mathbf{r}) W^p(\mathbf{r},\, t) u_k(\mathbf{r}). \tag{3.11}$$

We find its equation of motion by expanding Eq. (2.8), using Eq. (3.7), forming the integrals in Eq. (3.11), and using the SCA, with the result that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} W_{j,k}^p(t) = {} & \gamma_\parallel R_{j,k}^p - (\gamma_\parallel - i\Delta_p) W_{j,k}^p(t) \\ & + \rho \frac{1}{\tau} \int_{t-\tau}^t \mathrm{d}t' \exp(i\Delta_p t')\theta_{j,k}(t') \\ & - \frac{i}{4\hbar} \sum_{m,n} \alpha_{j,k,m,n} f_{m,p}^+ E_m^*(t) P_n^{m+p}(t) \\ & + \frac{i}{4\hbar} \sum_{m,n} \alpha_{j,k,n,m} f_{m,p}^{-*} E_m(t)[P_n^{m-p}(t)]^*,\end{aligned} \tag{3.12}$$

where $R_{j,k}^p$ and $\theta_{j,k}(t)$ are related to $R(\mathbf{r}, t)$ and $\theta(\mathbf{r}, t)$ by the same projection as is used on the inversion in Eq. (3.11) and are given by

$$R_{j,k}^p(t) \equiv \epsilon \int_G \mathrm{d}^3 r u_j^*(\mathbf{r}) R^p(\mathbf{r},\, t) u_k(\mathbf{r}), \tag{3.13}$$

where $R^p(\mathbf{r}, t)$ is related to $R(\mathbf{r}, t)$ as $P_k(t)$ is related to $P(\mathbf{r}, t)$ in Eq. (3.5) and

$$\theta_{j,k}(t) \equiv \epsilon \int_G \mathrm{d}^3 r u_j^*(\mathbf{r}) \theta(\mathbf{r},\, t) u_k(\mathbf{r}). \tag{3.14}$$

The coefficients

$$\alpha_{j,k,m,n} = \epsilon \int_G \mathrm{d}^3 r u_j^*(\mathbf{r}) u_k(\mathbf{r}) u_m^*(\mathbf{r}) u_n(\mathbf{r}) \tag{3.15}$$

represent the spatial overlap of the field modes in the gain medium. This is the term that couples the modes, resulting in competition effects such as spatial hole burning and NDFWM.

To find the equation of motion for the field, expand Eq. (2.9), using Eqs. (3.1) and (3.5), and apply the integral $\int_C d^3 r u_j^*(\mathbf{r})$, using Eqs. (3.2) and (3.3) and the SCA. This yields

$$\frac{d}{dt} E_q(t) = -\frac{1}{2} \kappa_q E_q(t) + \frac{i2\pi}{\epsilon} \omega_q P_q^q(t). \quad (3.16)$$

Obtaining Eq. (3.16) requires the use of the fact that $P(\mathbf{r}, t) = 0$ outside the gain medium and the assumption that the dielectric constant ϵ is constant within the gain medium. Equation (3.16) shows that each polarization mode drives a single field mode, as suggested by expansions (3.5) and (3.6). The loss rate for each mode is represented by

$$\kappa_q = \int_C d^3 r |u_q(\mathbf{r})|^2 \kappa(\mathbf{r}, \omega_q). \quad (3.17)$$

A simple expression for the distributed loss $\kappa(\mathbf{r}, \omega_q)$ that could apply to Fig. 1 when the cavity is filled by the gain medium (i.e., $L_G = L_C$) is $-c\epsilon(\mathbf{r})\ln[\Re_1(\omega_q)\Re_2 \times (\omega_q)]/2L_C$. In this case the integral of Eq. (3.17) is evaluated over the entire cavity, with the (one-dimensional) result that

$$\kappa_q = -c \ln[\Re_1(\omega_q)\Re_2(\omega_q)]/2L_C, \quad (3.18)$$

which is expected for a single cavity when the only loss mechanism is due to transmission through the cavity mirrors.

More generally, if the field mode function in the region of the cavity adjacent to the output coupler of reflectivity $\Re$ has the form $u_q(z) = A_q \sin(k_q z) = [\exp(ik_q z) - \exp(-ik_q z)]/2i$ [with $A_q = A(\omega_q)$], then $\kappa(\mathbf{r}, \omega_q) = cA_q^2[1 - \Re(\omega_q)]\epsilon(\mathbf{r})/4$, so[54]

$$\kappa_q = cA_q^2[1 - \Re(\omega_q)]/4. \quad (3.19)$$

Exact expressions for the spatial-mode amplitude A_q can be calculated for particular compound cavities.[54]

Equations (3.8), (3.12), and (3.16) form a general description of lasers in which the decay rates $\gamma_\perp$, $\gamma_\parallel$, and κ_q may have comparable magnitudes. They allow for a finite-sized gain region with an index that is different from that in the rest of the cavity, and they include quantum noise in a self-consistent manner. They are now specialized for specific classes of lasers by the classification scheme given by Arecchi *et al.*[60]: Lasers for which $\gamma_\perp \approx \gamma_\parallel \approx \kappa_q$ are class C; if $\gamma_\perp \gg \gamma_\parallel \approx \kappa_q$ the laser is class B; finally, class A lasers are ones for which $\gamma_\perp \gg \gamma_\parallel \gg \kappa_q$.

4. CLASS-B LASER EQUATIONS

In systems for which $\gamma_\perp \gg \gamma_\parallel$, κ_q (class-B lasers) and $\delta \ll \gamma_\perp$ the polarization can be adiabatically eliminated formally by integration of Eq. (3.8) over a time step τ long compared with $1/\gamma_\perp$. This is equivalent to using the self-consistent approximation that $dP_j^q(t)/dt \approx 0$ rather than the often used, but less reliable, assumption that $dP(\mathbf{r}, t)/dt \approx 0$.[14,32,61] The result is

$$P_j^q(t) \approx -\frac{i\mu^2}{\hbar\beta_q} \sum_p f_{q,p}^- E_{q-p}(t) W_{j,q-p}^p(t) + \frac{1}{\beta_q} Q_j^q(t). \quad (4.1)$$

Substituting relation (4.1) into Eq. (3.16) yields the class-B equation of motion for the field mode amplitudes:

$$\frac{d}{dt} E_j(t) = -{}^1\!/_2 \kappa_j E_j(t) + \frac{g_j^2}{\beta_j} \sum_p f_{j,p}^- W_{j,j-p}^p(t) E_{j-p}(t) + \sqrt{8\pi\hbar\omega_j} F_j^j(t), \quad (4.2)$$

where $g_j^2 = 2\pi\omega_j\mu^2/\hbar$ represents the field–matter coupling. Equation (4.2) is a generalization of an expression that is often used to model semiconductor lasers[13] in that the inversion spatial dependence is treated and the intrinsic inversion noise is included.

The field-mode noise $F_j^j(t)$ and part of the inversion noise in class-B lasers arise from the polarization noise. This noise is related to the polarization noise $Q_j^q(t)$, given in Eq. (3.10), by

$$F_j^q(t) = \frac{i}{\beta_j} (\pi\omega_j/2\hbar\epsilon)^{1/2} Q_j^q(t) \quad (4.3)$$

and is scaled so it represents the photon-number noise in the photon-number equations of motion that can be developed from the field equations, Eq. (4.2). The correlation function of this noise reads as

$$\langle F_j^{q*}(t') F_k^p(t)\rangle = \frac{2\gamma_\perp g_j g_k}{\beta_j^* \beta_k} f_{k,j}^- W_{k,j}^{p-q}(t)\delta(t' - t). \quad (4.4)$$

Note that only the diagonal, i.e., $j = q$ and $k = p$ in Eq. (4.4), noise terms play a role in the temporal evolution of the field modes in Eq. (4.2). The $W_{k,j}^{p-q}(t)$ term in Eq. (4.4) is a stochastic, complex, dynamic variable [as given by Eq. (3.12)] and is not generally zero when $j \neq k$ or $p \neq q$. Equation (4.4) therefore indicates that a correlation exists between the noises $F_j^j(t)$ and $F_k^k(t)$ of different mode amplitudes. This correlation has been pointed out by Marcuse[62] but without the analytic development presented here.

An estimate of the degree of correlation between mode noises can be found from an analysis of the relation between the coefficients $W_{j,k}^p(t)$ and the actual number of inverted atoms. Writing the inversion as a sum of the total inversion number $N(t)$ and the spatial variations $X(\mathbf{r}, t)$ about the mean $N(t)$ that are due to mode competition, we have

$$W(\mathbf{r}, t) = N(t)/V_G + X(\mathbf{r}, t), \quad (4.5)$$

where V_G is the volume of the active gain region and

$$N(t) = \int_G d^3 r W(\mathbf{r}, t). \quad (4.6)$$

This yields

$$W_{j,k}^p(t) = \delta_{j,k}\delta_{p,0} N(t)/V_C + X_{j,k}^p(t), \quad (4.7)$$

where $\delta_{j,k}$ is the Kronecker delta function and $X_{j,k}^{p}(t)$ is defined from $X(\mathbf{r}, t)$ similarly to Eq. (3.11). To obtain Eq. (4.7) we used Eq. (3.7), the fact that $N(t)$ is zero outside the gain medium at all times, $\lambda \ll L_G$, and $\gamma_\parallel \ll \bar{\Delta}$. Note that $X_{j,k\neq j}^{p}(t) = W_{j,k\neq j}^{p}(t)$. The $X_{j,k}^{p}(t)$ terms represent mode competition effects such as spatial hole burning ($p = 0$) and NDFWM ($p \neq 0$) and are not zero in general. Thus, according to Eq. (4.4), the individual mode noises are generally correlated by NDFWM, which includes the effects of coherent mode coupling and nonlinear gain. (The coherent mode coupling and the nonlinear gain terms are defined in the discussion of class-A laser equations below.) Furthermore, as is evident from Eq. (3.12), when the pump $R(\mathbf{r}, t)$ is not spatially uniform or has fast components the inversion components $X_{j,k}^{p\neq 0}(t)$ are pumped, making the mode-noise correlation more significant. However, when $\bar{\Delta} \gg \gamma_\parallel, \kappa_j$, as in most semiconductor laser systems, it can be seen that the high-frequency inversion components $X_{j,k}^{p\neq 0}$ that correlate the mode noises are small (of the order of $\gamma_\parallel/\bar{\Delta}$) compared with the average inversion density $N(t)$. Thus, in certain class-B lasers, such as the one considered in the companion paper[28] and in Refs. 25 and 26, the correlated mode noise is small compared with the uncorrelated noise, as pointed out by Marcuse.[62] This may not be the case for certain class-A lasers, discussed below.

In short-cavity systems the intermodal beat frequencies are large, so $\bar{\Delta} \gg \gamma_\parallel, \kappa_j$, resulting in the high-frequency components ($p \neq 0$) of the inversion being small and adiabatically following the field. Thus these components can also be eliminated adiabatically. Equation (3.12) thus gives an iterative expansion that can be solved perturbatively, yielding, in the SCA,

$$W_{j,k}^{p\neq 0}(t) \approx \frac{1}{(\gamma_\parallel - i\Delta_p)}\Bigg[\gamma_\parallel R_{j,k}^{p} + G_{j,k}^{p}(t) - \frac{\mu^2}{4\hbar^2}\sum_{m,n}\frac{\alpha_{j,k,m,n}}{\beta_{m+p}} f_{m,p}^{+} E_m^{*}(t) E_{m+p}(t) \times W_{n,m+p}^{0}(t) - \frac{\mu^2}{4\hbar^2}\sum_{m,n}\frac{\alpha_{j,k,n,m}}{\beta_{m-p}^{*}} f_{m,p}^{-*} E_m(t) \times E_{m-p}^{*}(t) W_{m-p,n}^{0}(t)\Bigg] \tag{4.8}$$

to second order in the small perturbation parameter $\xi = \mu^2\alpha|A|^2/4\hbar^2(\gamma_\parallel - i\Delta)(\gamma_\perp - i\Delta)$, where α is a typical value of the spatial overlap integral $\alpha_{j,k,m,n}$ and ranges from 0 to less than $3\epsilon/2V_C$, depending on the specific functional form of $u_j(\mathbf{r})$, the indices j, k, m, and n, and the geometry of the gain medium and the cavity.

Equation (3.12) (with $p = 0$), combined with Eqs. (4.1) (4.2), forms the class-B equations of motion for the field and inversion in a multimode laser. Rewritten, with the coherent terms ($p \neq$ sums) separated, these equations read as

$$\frac{\mathrm{d}}{\mathrm{d}t} E_j(t) = -\frac{1}{2}\kappa_j E_j(t) + \sqrt{8\pi\hbar\omega_j} F_j^{j}(t) + \frac{g_j^2}{\beta_j}\Bigg[W_{j,j}^{0}(t) E_j(t) + \sum_{p,p\neq 0} f_{j,p}^{-} W_{j,j-p}^{p}(t) E_{j-p}(t)\Bigg], \tag{4.9}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} W_{j,k}^{0}(t) = {}& \gamma_\parallel[R_{j,k}^{0} - W_{j,k}^{0}(t)] + G_{j,k}^{0}(t) \\ & - \frac{\mu^2}{4\hbar^2}\sum_{m,n}\frac{\alpha_{j,k,m,n}}{\beta_m}\Bigg[|E_m(t)|^2 W_{n,m}^{0}(t) + \sum_{p,p\neq 0} f_{m,p}^{-} E_m^{*}(t) E_{m-p}(t) W_{n,m-p}^{p}(t)\Bigg] \\ & - \frac{\mu^2}{4\hbar^2}\sum_{m,n}\frac{\alpha_{j,k,n,m}}{\beta_m^{*}}\Bigg[|E_m(t)|^2 W_{m,n}^{0}(t) + \sum_{p,p\neq 0} f_{m,p}^{-*} E_{m-p}^{*}(t) E_m(t) W_{n,m-p}^{p*}(t)\Bigg],\end{aligned} \tag{4.10}$$

where we evaluate the $W_{j,k}^{p\neq 0}(t)$ terms by using relation (4.8). Note that the sums (over m, n, p) in Eq. (4.10) are complex conjugates if the mode functions $u_j(\mathbf{r})$ are real (e.g., standing waves).

We now find the correlation function of the spontaneous-emission noise (which is due to the polarization in the SCA) by combining Eqs. (4.4) and (4.7):

$$\langle F_j^{j*}(t')F_k^{k}(t)\rangle = \frac{2\gamma_\perp g_j g_k}{\beta_j^{*}\beta_k}\left[\frac{N(t)}{V_C}\delta_{j,k} + f_{k,j}^{-} X_{k,j}^{k-j}(t)\right] \times \delta(t' - t). \tag{4.11}$$

Equation (4.11) confirms that it is the high-frequency ($j \neq k$) inversion terms that result in the field-mode noise correlations noted with regard to Eq. (4.4).

The inversion noise in relation (4.8) is described (in the SCA) by

$$G_{j,k}^{p}(t) = \frac{\rho}{\tau}\int_{t-\tau}^{t}\mathrm{d}t' \exp(i\Delta_p t')\theta_{j,k}(t') - \frac{\mu}{2\hbar}\sum_{m,n}\left[\frac{\alpha_{j,k,m,n}}{g_{m+p}} f_{m,p}^{+} E_m^{*}(t) F_n^{m-p}(t) + \frac{\alpha_{j,k,n,m}}{g_{m-p}} f_{m,p}^{-*} E_m(t) F_n^{m-p*}(t)\right] \tag{4.12}$$

and has zero mean.[49] The first term in Eq. (4.12) is the intrinsic inversion noise given by Eq. (3.14). The second term is due to the eliminated polarization noise.

The $W_{j,k\neq j}^{0}(t)$ terms in Eq. (4.10) may be important to the dynamics of $W_{j,j}^{0}$, when the pump is not spatially uniform, because in that case they are pumped ($R_{j,k\neq j}^{0} > 0$). However, as is evident from Eq. (4.7), they never contribute to the number of excited lasing centers $N(t)$, given by Eq. (4.6), and thus to the total system energy. The stochastic part of the fluctuations in the total intensity can arise mainly from $G_{j,k}^{0}(t)$ noise terms. The high-frequency inversion components, $W^{p\neq 0}_{j,j-p}(t)$, represent the coherent coupling of modes (the part that depends on the relative phases of the field components) as well as on a small piece of the incoherent cross saturation (i.e., the nonlinear gain in semiconductor lasers).

The requirement that $|\xi| \ll 1$ for the iterative expansion of $W^{p\neq 0}_{j,j-p}(t)$ in relation (4.8) reduces to the condition that the fraction above lasing threshold be much less than $4\bar{\Delta}/\gamma_{\parallel}$. Thus, for a wide range of systems, the coherent coupling terms that result from the perturbation expansion are valid for pump rates several times above threshold.

Expressions (4.8)–(4.12) are the main result of this paper. They model multimode lasers where $\gamma_{\perp} \gg \bar{\Delta} \gg \gamma_{\parallel}, \kappa_j$ and are in a form that is numerically tractable. They incorporate relaxation oscillations through the $W^{0}_{j,j}(t)$ terms as well as coherent coupling from the $W^{p\neq 0}_{j,j-p}(t)$ terms and allow for spatially nonuniform pumping with no assumptions regarding the functional form of the population-inversion spatial dependence. The noise terms have been derived from an amplifier model that is consistent with quantum mechanics, excluding only so-called nonclassical effects such as squeezing and sub-Poissonian statistics.

5. CLASS-A LASER EQUATIONS

The correlation of the mode noises in Eq. (4.4) is most pronounced in systems in which the coherent mode coupling is large. This coupling is important in well-known class-A laser systems for which $\gamma_{\perp} \gg \gamma_{\parallel} \gg \kappa_j$ (see, e.g., Refs. 9 and 14). In these lasers, not only can the high-frequency inversion components $W^{p\neq 0}_{j,k}(t)$ be eliminated but the slow parts $W^{0}_{j,k}(t)$, given by Eq. (4.10), can be eliminated as well. We can find the relevant terms by setting the left-hand side of Eq. (4.10) to zero. We solve the resulting iterative solution to first order by substituting the zeroth-order solution, $W^{0}_{j,k}(t) \approx R^{0}_{j,k}$, on the right-hand side. The relevant result is

$$W^{0}_{j,j}(t) \approx R^{0}_{j,j} - \frac{\mu^2}{2\hbar^2\gamma_{\parallel}} \sum_{m,n} |E_m(t)|^2 \,\mathrm{Re}\!\left(\frac{\alpha_{j,j,m,n}}{\beta_m} R^{0}_{n,m}\right) + \frac{G^{0}_{j,j}(t)}{\gamma_{\parallel}}. \tag{5.1}$$

Substituting relation (4.8) and Eq. (5.1) into Eq. (4.9) yields third-order multimode field equations that include the effects of line shape, spatial hole burning, four-wave mixing, and spatially nonuniform pump and polarization noise (including correlated noises), valid for running- or standing-wave field modes.

In a laser for which the pump is spatially uniform and the field modes can be treated as plane waves, e.g., $u_j(z) = \sqrt{2/L_C}\,\sin(j\pi z/L_C)$, the inversion terms can be written as

$$W^{0}_{j,j}(t) \approx R\,\frac{L_G}{L_C}\left[1 - \frac{\mu^2}{2\hbar^2\gamma_{\parallel}\gamma_{\perp}} \sum_{m} \alpha_{j,j,m,m} l_m |E_m(t)|^2\right] + \frac{G^{0}_{j,j}(t)}{\gamma_{\parallel}}, \tag{5.2}$$

$$\begin{aligned} W^{p\neq 0}_{j,j-p}(t) \approx{}& \frac{\mu^2 R}{4\hbar^2\gamma_{\perp}(\gamma_{\parallel} - i\Delta_p)}\,\frac{L_G}{L_C}\Bigg[\alpha_{j,j-p,j-p,j} \times \left(\frac{1}{\beta_j} + \frac{1}{\beta^{*}_{j-p}}\right) E^{*}_{j-p}(t) E_j(t) \\ &+ \sum_{m,m\neq j-p} \frac{\alpha_{j,j-p,m,m+p}}{\beta_{m+p}} E^{*}_{m}(t) E_{m-p}(t) \\ &+ \sum_{m,m\neq j} \frac{\alpha_{j,j-p,m-p,m}}{\beta^{*}_{m-p}} E_m(t) E^{*}_{m-p}(t)\Bigg] \\ &+ \frac{G^{p}_{j,j-p}}{\gamma_{\parallel} - i\Delta_p}, \end{aligned} \tag{5.3}$$

where

$$l_j = \gamma_{\perp}\,\mathrm{Re}(1/\beta_j) = \gamma_{\perp}^2/[\gamma_{\perp}^2 + (\delta - \Delta_j)^2] \tag{5.4}$$

represents the gain line shape of the laser. Substituting relations (5.2) and (5.3) into Eq. (4.9) yields a third-order multimode field equation that describes typical dye lasers ($L_C = 30$ cm) that are known to exhibit irregular (possibly chaotic) behavior.[14,58,63] The first term in relation (5.2) yields the linear gain, and the sum in this equation contributes the self-saturation $(m = j)$ and cross-saturation $(m \neq j)$ terms to field equation (4.9). The first term in brackets of relation (5.3) yields incoherent (i.e., no dependence on the phase of the field) terms of the same form as the cross-saturation terms that result from relation (5.2) but much smaller in magnitude as long as $\bar{\Delta} \gg \gamma_{\parallel}$. These terms are often labeled the nonlinear gain in the semiconductor laser literature.[44,64] The remaining contribution of relation (5.3) consists of the coherent field-mode coupling (NDFWM) that is due to the two sums and the noise $G^{p}_{j,j-p}(t)$. The noise contributions of relations (5.2) and (5.3) are generally neglected in class-A systems.[65]

Note that the equations presented here generalize those given by McMackin *et al.*[14] and Beck *et al.*[58] in that they include the effects of gain line shape, nonuniform mode spacing, and correlated field-mode noise. Inclusion of this field-mode noise is consistent with the inclusion of coherent coupling (NDFWM) that has been shown to be the dominant source of the mode fluctuations in certain multimode dye lasers.[14] In cases like this, in which coherent coupling effects are significant, the correlated noise is not necessarily small compared with the uncorrelated noise, as it is in the class-B case considered above.

6. CONCLUSION

We have presented a laser model that self-consistently includes population dynamics, including coherent mode coupling, as well as quantum-mechanical noise and local-field corrections within the context of a four-level gain medium approximated with two levels. We make no assumptions regarding the explicit spatial dependence of the inversion or the pump. The model yields familiar results for simple cavities, in which the gain medium fills the cavity, and also describes (strongly) coupled-cavity systems, given that the spatial electric-field mode functions are known.[54]

In the companion paper,[28] we apply the class-B laser equations developed here to simulate a short-cavity dye laser.[25,26] The good agreement that we find between the measured and the modeled turn-on time statistics depends on the simulation including the intrinsic population noise.

Our analysis indicates that the individual mode noises are correlated through the influence of coherent mode coupling, contrary to what is often assumed. Although the effects of coherent mode coupling (i.e., four-wave mixing) appear to be insignificant in the turn-on transient dynamics and statistics discussed in the companion paper and in Ref. 25, they may well play an important role in systems in which coherent coupling is significant (see, e.g., Refs. 14 and 58).

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