Limits to wideband pulsed squeezing in a traveling-wave parametric amplifier with group-velocity dispersion

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A theoretical method is developed to treat wideband pulsed squeezing in a traveling-wave parametric amplifier with group-velocity dispersion. Classical stochastic wave equations that are fully equivalent to operator equations of motion are developed and solved numerically. It is found that squeezing occurs over the entire phase-matching bandwidth, although the degree of squeezing decreases when the pump-pulse duration is shorter than the inverse of this bandwidth.

After the first demonstrations of cw quadrature squeezing of light fields by means of four-wave mixing,[1] parametric oscillation,[2] and the Kerr effect,[3] attention turned to experiments that used pulsed interactions as a possible method to obtain wideband squeezing with large numbers of photons per mode.[4–7] An important limitation to efficient wideband squeezing by pulsed, traveling-wave optical parametric amplification[4–6] is group-velocity dispersion (GVD) of the subharmonic field in the nonlinear medium, which imposes a finite bandwidth over which phase matching can be achieved. In the case of a monochromatic cw pump laser, the effects of dispersion on parametric amplification are known.[8–10] In this case pairs of modes, whose frequencies sum to the pump frequency, are generated independently of one another. The wideband theory is then a straightforward extension of the two-mode theory. On the other hand, when the pump field has nonzero bandwidth, as in the case of a pulse, all downconverted modes couple to each other through the different spectral components of the pump, which makes the problem more complex. The case of pulsed parametric amplification without GVD has been treated by solving for the Heisenberg-picture field operators in the space–time domain.[11] Without GVD an essentially infinite bandwidth of squeezing is predicted.

This Letter presents a theory of wideband squeezing by traveling-wave parametric amplification that includes GVD and an arbitrary input pump field, with emphasis on the case of a pulsed pump. The approach uses the method of the positive-P representation[12] to generate a set of stochastic, c-number equations of motion that are fully equivalent to the quantum-mechanical Heisenberg-picture equations for the field operators. The c-number equations thus can describe nonclassical fields, such as that generated in squeezing experiments, while also allowing for nontrivial dynamics of the fields, such as pump depletion or pulse delay and spreading due to the linear medium response.

The present treatment is based on a macroscopic approach to quantizing the electromagnetic field in a nonlinear dielectric medium with GVD, developed only recently.[13,14] A Lagrangian that generates both the correct equations of motion and total energy underlies this theory.[13] As such, it is free from the nonuniqueness and inconsistencies that can occur in *ad hoc* quantization schemes, as pointed out previously.[15]

The Hamiltonian for a dispersive $\chi^{(2)}$ medium is found from the Lagrangian formalism to be, up to third order in the field, in S.I. units,[13]

$$\hat{H} = \sum_m \hbar\omega_m^{(1)}\hat{a}_m^{(1)\dagger}\hat{a}_m^{(1)} + \sum_m \hbar\omega_m^{(2)}\hat{a}_m^{(2)\dagger}\hat{a}_m^{(2)} - \frac{1}{3}\varepsilon_0\chi^{(2)}\int d^3x\left[\frac{D^{(1)}(\mathbf{x})}{\varepsilon_1} + \frac{D^{(2)}(\mathbf{x})}{\varepsilon_2}\right]^3. \quad (1)$$

Here $D^{(1)}(\mathbf{x})$ and $D^{(2)}(\mathbf{x})$ are electric displacement fields, assumed scalar, oscillating near optical frequencies ω_1 (subharmonic) and ω_2 (fundamental), respectively. The bandwidths of these fields are assumed to be narrow compared to ω_1 and ω_2. The dielectric constants at these two frequencies are ε_1 and ε_2. The proper fields to quantize in a dielectric are the vector potential $A(\mathbf{x})$ and the electric displacement $D(\mathbf{x})$ (not the electric field as is sometimes assumed). For simplicity we specialize to one-dimensional propagation, as would occur in a waveguide or in free space with a unity-Fresnel-number pumped volume. In a dispersive dielectric medium with group velocity equal to $d\omega/dk|_{\omega=\omega_i} \equiv \omega_i'$ at frequency ω_i, the electric displacement near this frequency is expanded in terms of boson operators $\hat{a}_m^{(i)}$ and $\hat{a}_m^{(i)\dagger}$ and normalized transverse spatial modes $u^{(i)}(\mathbf{x}_T)$ as[13]

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$$\hat{D}^{(i)}(\mathbf{x}) = i\sum_m \left[\frac{\varepsilon_i \hbar \omega_i' k_m^{(i)}}{2L}\right]^{1/2} \hat{a}_m^{(i)} u^{(i)}(\mathbf{x}_T)\exp[ik_m^{(i)}z] + h.a. \quad (i = 1, 2), \quad (2)$$

where the propagation constants are $k_m^{(i)} = n(\omega_i)\omega_i/c + m\Delta k$, with $n(\omega_i) = (\varepsilon_i/\varepsilon_0)^{1/2}$ the refractive index, m an integer, $\Delta k = 2\pi/L$, and L the longitudinal dimension of the medium (quantization volume). Phase matching $[k_0^{(2)} = 2k_0^{(1)}]$ is assumed at frequency ω_1. The mode frequencies $\omega_m^{(i)}$ in Eq. (1) are dependent on the linear dispersion and are expanded to second order in $m\Delta k$ so that

$$\omega_m^{(i)} \cong \omega_i + m\Delta k\omega_i' + \frac{1}{2}(m\Delta k)^2\omega_i'', \quad (3)$$

where $\omega_i'' \equiv d^2\omega/dk^2|_{\omega=\omega_i}$ leads to GVD near each frequency ω_i $(i = 1, 2)$.

Following the method of Ref. 16, spatially localized operators $\hat{\alpha}_l$ and $\hat{\beta}_l$ for both fields are defined on a one-dimensional grid with $2M + 1$ points by

$$\hat{\alpha}_l = \frac{1}{(2M+1)^{1/2}}\sum_{m=-M}^{M} \hat{a}_m^{(1)} \exp\left(\frac{i2\pi ml}{2M+1}\right) \quad (4)$$

and a similar expression relating $\hat{\beta}_l$ to $\hat{a}_m^{(2)}$, where the integer l labels the spatial coordinate in the longitudinal direction z, i.e., $z_l = l\Delta z = lL/(2M + 1)$. The third, nonlinear, part of the Hamiltonian in Eq. (1) can be written in terms of these variables as

$$H_{\mathrm{NL}} \cong \frac{1}{2} i\hbar\tilde{\chi}\omega_1'(\omega_2'/\Delta z)^{1/2}\sum_l \hat{\alpha}_l^{\dagger 2}\hat{\beta}_l + h.a. \quad (5)$$

The rotating-wave approximation has been used, and the coupling constant $\tilde{\chi}$ is given by

$$\tilde{\chi} = \frac{\varepsilon_0\chi^{(2)}k_0^{(1)}}{\varepsilon_1}\left[\frac{\hbar k_0^{(2)}}{2\varepsilon_2}\right]^{1/2}\int d^2x_T[u^{(1)*}(\mathbf{x}_T)]^2 u^{(2)}(\mathbf{x}_T). \quad (6)$$

The linear parts of the interaction Hamiltonian can also be written in terms of $\hat{\alpha}_l$ and correspond to spatial (z) derivatives of $\hat{\alpha}_l$ in the continuum limit $(M \to \infty)$.[16]

Rather than deriving Heisenberg equations of motion for the field operators, which are difficult to solve, we can employ the positive-P representation to generate equations of motion for exactly equivalent c-number, stochastic variables α_l, α_l^+, β_l, and β_l^+ (coherent-state amplitudes) that correspond to operators $\hat{\alpha}_l$, $\hat{\alpha}_l^\dagger$, $\hat{\beta}_l$, and $\hat{\beta}_l^\dagger$, respectively.[12,17] This assumes vanishing boundary conditions for the P distribution. In this representation α_l and α_l^+ (as well as β_l and β_l^+) are not complex conjugates, so a doubling of the size of the phase space occurs in this method for describing the nonclassical system by c-number variables. The stochastic equations will be presented here in the continuum limit, in terms of c-number fields, defined by

$$\Phi(z_l) = (\omega_1'/\Delta z)^{1/2}\alpha_l,$$
$$\Psi(z_l) = (\omega_2'/\Delta z)^{1/2}\beta_l, \quad (7)$$

and similar relations connecting α_l^+ with $\Phi^+(z_l)$ and β_l^+ with $\Psi^+(z_l)$. The product $\Phi^+(z_l)\Phi(z_l)$ $[\Psi^+(z_l)\Psi(z_l)]$ corresponds to the number of subharmonic (fundamental) photons per second transported across the z_l plane. The stochastic equations, written in the frame moving at the group velocity ω_1' of the subharmonic, are found, by following a procedure similar to that in Ref. 16, to be

$$\left(\frac{\partial}{\partial z} + \frac{i}{2}k_1''\frac{\partial^2}{\partial\tau^2}\right)\Phi(z,\tau) = \tilde{\chi}\Psi(z,\tau)\Phi^+(z,\tau) + [\tilde{\chi}\Psi(z,\tau)]^{1/2}\Gamma(z,\tau), \quad (8a)$$

$$\left[\frac{\partial}{\partial z} + \left(\frac{1}{\omega_2'} - \frac{1}{\omega_1'}\right)\frac{\partial}{\partial\tau} + \frac{i}{2}k_2''\left(\frac{\omega_2'}{\omega_1'}\right)^2\frac{\partial^2}{\partial\tau^2}\right]\Psi(z,\tau) = -\frac{1}{2}\tilde{\chi}^*\Phi^2(z,\tau), \quad (8b)$$

where $\tau = t - z/\omega_1'$, $k_i'' = d^2k/d\omega^2|_{k=k_i} = -\omega_i''/(\omega_i')^3$, and $\tilde{\chi}$ is given by Eq. (6). Equations for $\Phi^+(z,\tau)$ and $\Psi^+(z,\tau)$ are obtained by interchanging $\Phi \leftrightarrow \Phi^+$, $\Psi \leftrightarrow \Psi^+$, and $i \leftrightarrow -i$ and replacing $\Gamma(z,\tau)$ by $\Gamma^+(z,\tau)$ in Eqs. (7) and (8). The derivatives $\partial^2/\partial z^2$ and $\partial^2/\partial\tau\partial z$ were neglected, by analogy with semiclassical derivations of the nonlinear Schrödinger equation in $\chi^{(3)}$ media (for pulse lengths less than the gain length).[18] In this model the frequency dependence of the phase-matching properties is contained in the GVD constants k_1'' and k_2''. The stochastic Langevin terms $\Gamma(z,\tau)$ and $\Gamma^+(z,\tau)$ are real, independent, Gaussian processes with correlation functions

$$\langle\Gamma(z,\tau)\Gamma(z',\tau')\rangle = \langle\Gamma^+(z,\tau)\Gamma^+(z',\tau')\rangle = \delta(z - z')\delta(\tau - \tau'). \quad (9)$$

The problem can be put into dimensionless form by scaling the z variable by the gain length $z_0 = |\tilde{\chi}\Psi_0|^{-1}$ and the τ variable by the inverse phase-matching bandwidth $\tau_0 = \sqrt{z_0 k_1''}$. These depend on the peak value of the input field Ψ_0. Here we give examples only in the limit of an intense, classical pump field $\Psi(z,\tau) = \Psi_0\,\mathrm{sech}^2(\tau/\tau_p)$ that is unaffected by propagation through the medium. Relaxation of this assumption to include pump GVD, depletion, and quantum noise is easily treated by our methods. The quantum-statistical properties of the Φ field are determined by stochastic simulation of Eqs. (8). We treat the case of the squeezed vacuum, for which the input amplitudes $\Phi(0,\tau)$ and $\Phi^+(0,\tau)$ are zero. One thousand trajectories are obtained by numerically solving Eqs. (8) by using the central-difference method.[19] Statistical moments of $\Phi(z,\tau)$ and $\Phi^+(z,\tau)$ are found by averaging over the ensemble of trajectories. We find no diverging trajectories, which indicates that boundary terms of the P distribution do indeed vanish.

The quantity of interest is the spectrum of pulsed squeezing, defined with respect to a pulsed local oscillator field $\Psi_{\mathrm{LO}}(\tau)$, taken to be equal to $[\Psi(\tau, z)]^{1/2}$, as would be typical in experiments.[4–6] In the case of the squeezed vacuum, where $\langle\Phi(z,\tau)\rangle = 0$, and with the use of the correspondence between normally ordered operator expectations and the moments of the stochastic variables, the spectrum of pulsed squeezing is defined to be[20]

$$S(z,\omega) = \frac{2\pi\langle\tilde{x}(z,-\omega)\tilde{x}(z,\omega)\rangle}{\int|\Psi_{\mathrm{LO}}(\tau)|^2 d\tau}, \quad (10)$$

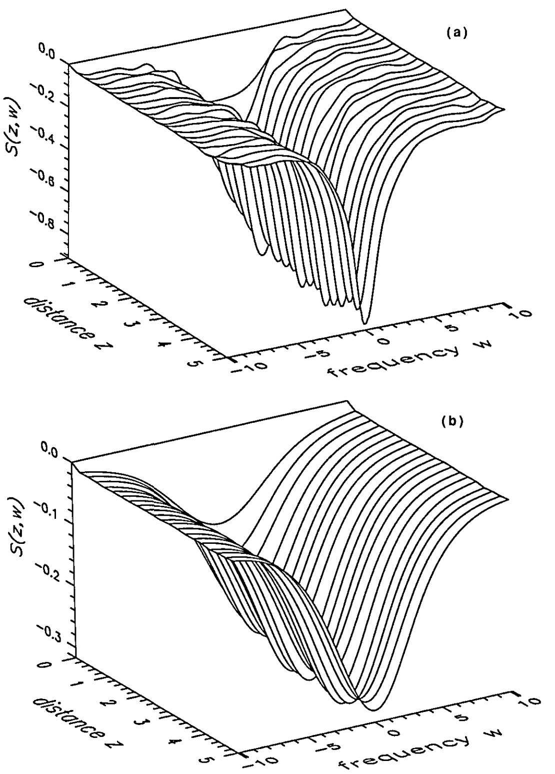

Fig. 1. Spectrum of pulsed squeezing as a function of distance (scaled by gain length) and frequency (scaled by phase-matching bandwidth) for two values of the pump-pulse duration τ_p: (a) $\tau_p = \tau_0$ (equals the inverse of the phase-matching bandwidth), (b) $\tau_p = \tau_0/5$. Wideband squeezing is seen over the entire phase-matching bandwidth, although the degree of squeezing is reduced for the shorter pulse.

where

$$x(z,\tau) = \eta\Psi_{\mathrm{LO}}^*(\tau)\Phi(z,\tau) + \eta^*\Psi_{\mathrm{LO}}(\tau)\Phi^+(z,\tau),$$

$$\tilde{x}(z,\omega) = \frac{1}{\sqrt{2\pi}}\int x(z,\tau)\exp(i\omega\tau)\,\mathrm{d}\tau, \tag{11}$$

and $\eta = |\eta|e^{i\theta}$ represents the beam-splitter amplitude for transmission of the signal field $\Phi(z,\tau)$. $S = 0$ corresponds to the shot-noise limit, and the lower bound on $S(z,\omega)$ is -1, at which point there is no noise in a balanced homodyne measurement. $S(z,\omega)$ is calculated by averaging over the ensemble of $\Phi(z,\tau)$ trajectories, with the beam-splitter phase θ adjusted independently for each frequency to give minimum $S(z,\omega)$.

Figure 1 shows how the spectrum of squeezing develops during propagation for two values of pump duration τ_p. Since the cw limit has been treated elsewhere,[9,10] we show results for pump pulses with durations of the order of the coherence time (inverse phase-matching bandwidth) ($\tau_p = \tau_0$) and significantly less than the coherence time ($\tau_p = \tau_0/5$). The frequency axis is scaled by the phase-matching bandwidth τ_0^{-1} ($\approx 3 \times 10^{14}$ rad/s for a typical crystal with a 1-cm gain length). The optical power spectrum (not shown) has a full width of approximately $2\tau_0^{-1}$ in both cases. Figure 1(a) shows strong squeezing over the entire optical spectral bandwidth for a propagation distance of five gain lengths. Figure 1(b) shows a large reduction in squeezing when the input pulse bandwidth is increased beyond the phase-matching bandwidth of the medium. This occurs because the input pulse frequency components are spread over too large a range for all the components to contribute to the coherent downconversion process. The cause of the oscillations as a function of distance (not due to numerical error) is unknown.

The cases of pump depletion, pump walk-off, and dispersion can be easily treated by our methods. We note that the rate of pump depletion is dependent on the downconversion bandwidth and therefore requires an understanding of dispersion. Our equations can also be generalized to include phase mismatch and three-dimensional diffraction effects. The equations are well suited to direct numerical simulation, without requiring the semiclassical limit for their validity.

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